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Exercise (Modeling with Different Types of Functions)

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1 Different Types of Functions 不同類型的函數

Different functions describe relations between input and output data. If we can describe the data with a function, we can start to discuss the interpretation of the function itself, or even predict the output data using the function. Therefore, functions are one of the most important parts of mathematical modelling.
不同的函數描述了輸入數據與輸出數據之間的關係。如果我們能夠用函數來描述數據，就可以開始討論函數本身的詮釋，甚至利用函數預測輸出數據。因此，函數是數學建模中最重要的部分之一。

The following are some common but important functions in mathematical modeling:
以下是一些在數學建模中常見且重要的函數：

Polynomial Functions 多項式函數

Polynomial functions have the following general form:
多項式函數具有以下一般形式：

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants.
其中$a_0, a_1, \ldots, a_n$ 均為常數。

For instance,
例如：

- When $n = 1$, we have a linear function $y = ax + b$.
  當$n = 1$ 時，為線性函數$y = ax + b$。
- When $n = 2$, we have a quadratic function $y = ax^2 + bx + c$.
  當$n = 2$ 時，為二次函數$y = ax^2 + bx + c$。
- When $n = 3$, we have a cubic function $y = ax^3 + bx^2 + cx + d$.
  當$n = 3$ 時，為三次函數$y = ax^3 + bx^2 + cx + d$。

- When $n = 4$, we have a quadric function $y = ax^4 + bx^3 + cx^2 + dx + e$.
  當$n = 4$ 時，為四次函數$y = ax^4 + bx^3 + cx^2 + dx + e$。

- When $n = 5$, we have a quintic function $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.
  當$n = 5$ 時，為五次函數$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$。

Our Linear and Non-linear Regression R Shiny tools (https://www.math.cuhk.edu.hk/app/mathmodel/tool.html) can be used for finding the best-fit polynomial functions for any given set of data points.
我們的線性與非線性迴歸R Shiny 工具(https://www.math.cuhk.edu.hk/app/mathmodel/tool.html) 可用於尋找任何給定數據點集的最佳擬合多項式函數。

Remark: Note that if we have n data points, we can always find a polynomial with degree $(n-1)$ passing through all points. However, we should be aware of the problem of overfitting!
註：若我們有n 個數據點，總能找到一個$(n-1)$ 次多項式穿過所有點。然而，我們需要注意過度擬合的問題！

Exponential Functions
指數函數

Exponential functions have the form:
指數函數的形式為：

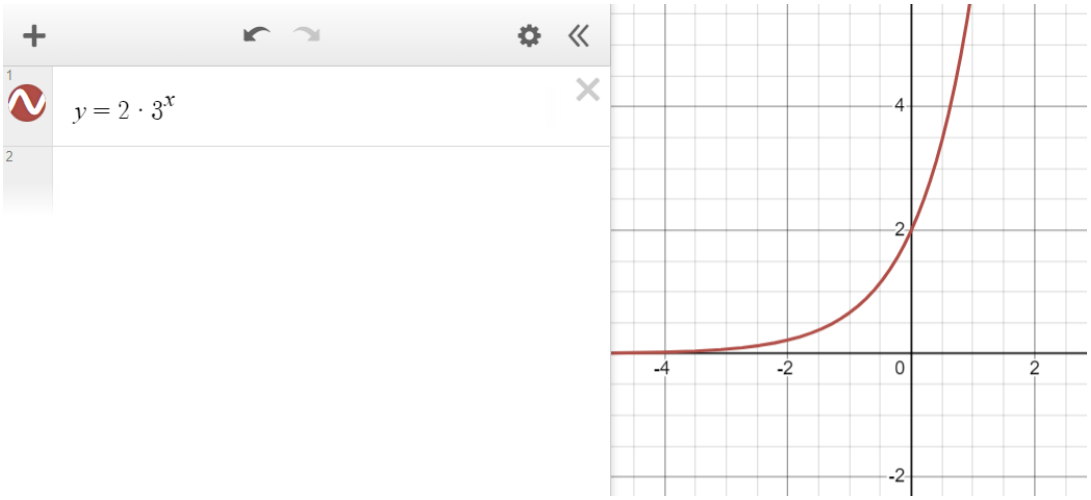
$$y = ab^x$$

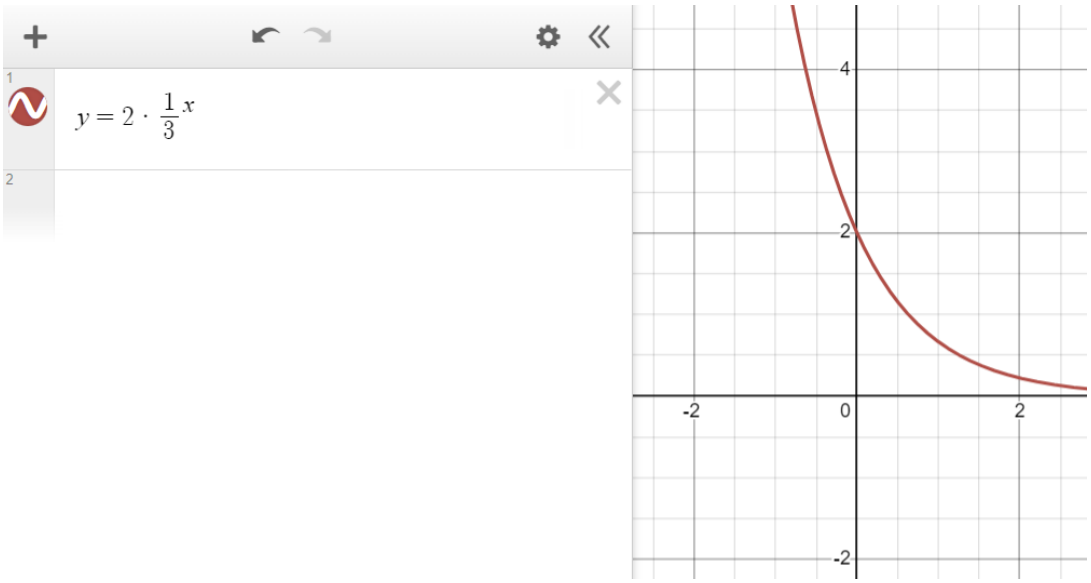
where
其中：

- a is initial value ($y = a$ when $x = 0$),
  a 為初始值（當$x = 0$ 時，$y = a$）

- b is the base of the exponential function, a positive real number. When $b > 1$, the function represents exponential growth; $0 < b < 1$ represents exponential decay.
  b 為指數函數的底數，是一個正實數。當$b > 1$ 時，函數表示指數增長；當$0 < b < 1$ 時，表示指數衰減。

Exponential Functions usually describe some rapid growth or decay. Therefore, sometimes we replace x with variable t, to indicate the change of time. The following are two graphs of the exponential functions:
指數函數通常描述某種快速增長或衰減的現象。因此，有時我們會用變量t 取代x，以表示時間的變化。以下是兩個指數函數的圖像：



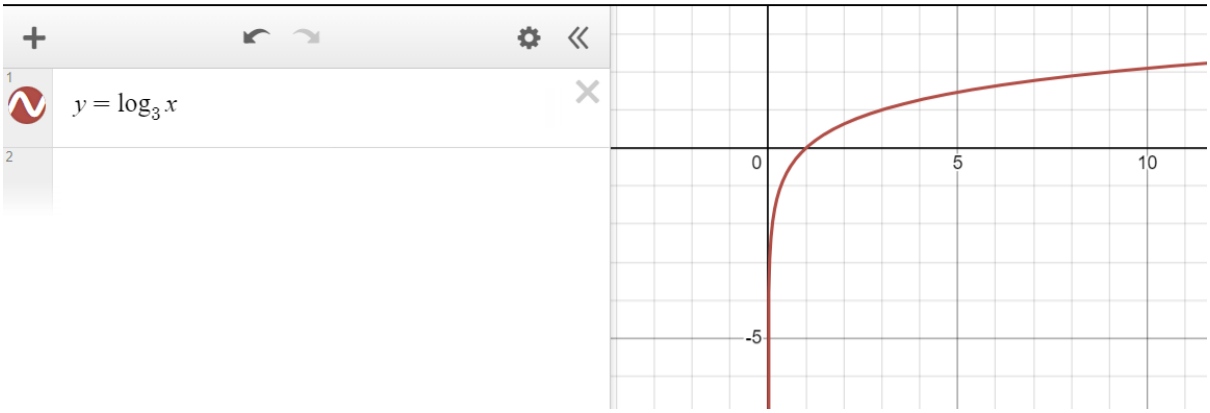


As we have mentioned, exponential functions demonstrate rapid growth or decay, according to the base.
正如我們所提及，指數函數根據底數的不同，表現出快速增長或快速衰減的特性。

Logarithmic Functions
對數函數

Logarithmic functions are the inverse of exponential functions. They 'undo' what exponential functions do and, therefore are used for turning exponential functions into linear functions. Instead of rapidly increasing, it shows a slowly increasing trend (increasing but decelerating rate).
對數函數是指數函數的反函數。它們「逆轉」了指數函數的作用，因此常用於將指數函數轉化為線性函數。對數函數並非快速增長，而是呈現緩慢增長的趨勢（增長但增長率遞減）。

Logarithmic functions have the form of $y = \log_b x$, and the graph looks like the following:
對數函數的形式為$y = \log_b x$，其圖像如下所示：



Please note that logarithmic functions have the following properties:
請注意，對數函數具有以下性質：

1. **Product Rule**
   積的法則
$$\log_b(xy) = \log_b(x) + \log_b(y)$$
2. **Quotient Rule**
   商的法則
$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$
3. **Power Rule**
   冪的法則
$$\log_b(x^k) = k \cdot \log_b(x)$$

Using the properties, we can turn exponential functions into linear functions as follows:
利用這些性質，我們可以將指數函數轉化為線性函數，如下所示：

$$y = a \cdot b^x$$

To linearize this function, we take the logarithm of both sides:
為將此函數線性化，我們對兩邊取對數：

$$\log(y) = \log(a \cdot b^x)$$

$$\log(y) = \log(a) + \log(b^x)$$
$$\log(y) = \log(a) + x \cdot \log(b)$$
$$\log(y) = x \cdot \log(b) + \log(a)$$

Here, $\log(y)$ is the dependent variable, x is the independent variable, $\log(b)$ is the slope, and $\log(a)$ is the intercept.
這裡，$\log(y)$ 為應變量，x 為自變量，$\log(b)$ 為斜率，$\log(a)$ 為截距。

Power Functions
冪函數

Power functions are in the form of
冪函數的形式為

$$y = a \cdot x^b$$

where
其中：

- a is a constant coefficient.
  a 為常數係數
- x is the variable.
  x 為變量
- b is a constant exponent.
  b 為常數指數

Power functions are more commonly found in mathematics and science. For example, the volume of spheres has the formula: $V = \frac{4}{3}\pi r^3$. This can be considered as a power function of the radius r.
冪函數在數學和科學中較為常見。例如，球體的體積公式為$V = \frac{4}{3}\pi r^3$，這可視為關於半徑r 的冪函數。

Please note that the sign of constant exponent b will greatly affect the shapes of graphs of the power functions. Check the following examples:
請注意，常數指數b 的正負號會大大影響冪函數圖像的形狀。請看以下例子：

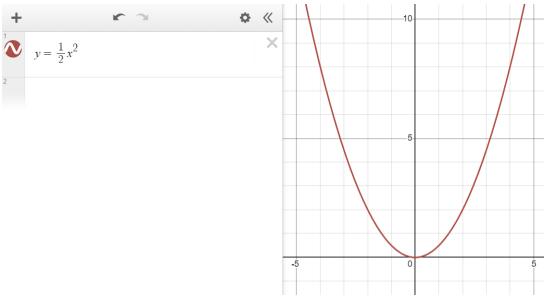


Figure 1: $\frac{1}{2}x^2$

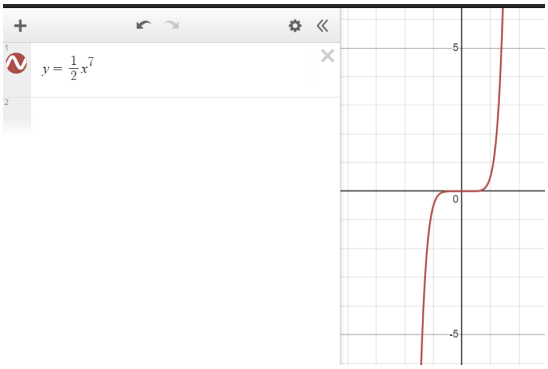


Figure 2: $\frac{1}{2}x^7$

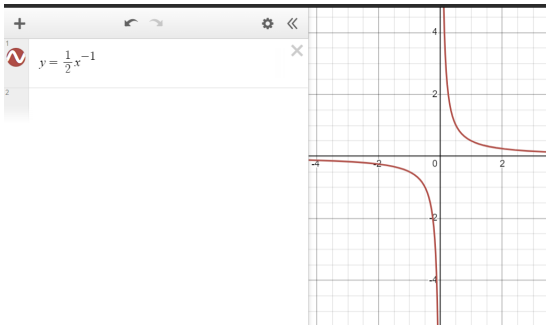


Figure 3: $\frac{1}{2}x^{-1}$

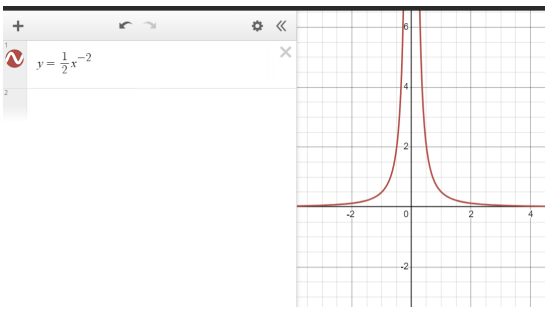


Figure 4: $\frac{1}{2}x^{-2}$

Trigonometric Functions
三角函數

Trigonometric functions are functions that relate the angles of a triangle to the lengths of its sides. They are fundamental in the study of geometry, especially in the context of right-angled triangles and the unit circle. They have the following form:
三角函數是將三角形的角與其邊長聯繫起來的函數。它們是幾何學研究的基礎，特別是在直角三角形和單位圓的情境中。其形式如下：

1. **Sine** (sin):
$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$
2. **Cosine** (cos):
$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
3. **Tangent** (tan):
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

However, in the context of mathematical modelling, we are more concerned about their periodic properties.
然而，在數學建模的語境中，我們更關注它們的週期性質。

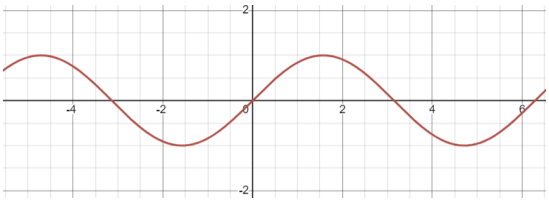


Figure 5: $y = \sin(x)$

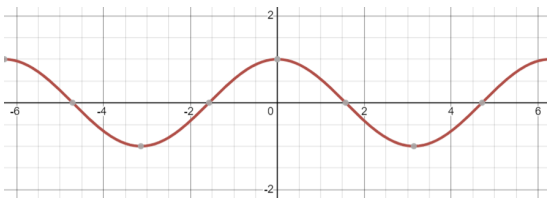


Figure 6: $y = \cos(x)$

With this special property, many things that are periodic can be described using trigonometric functions. From celestial movements in the sky to the seismic wave under the ground or even the electrocardiogram signals of our heart, they can be approximated with trigonometric functions!
憑藉這一特殊性質，許多週期性現象都可以用三角函數來描述。從天體運動到地底下的地震波，甚至我們的心電圖信號，都可以用三角函數來近似！

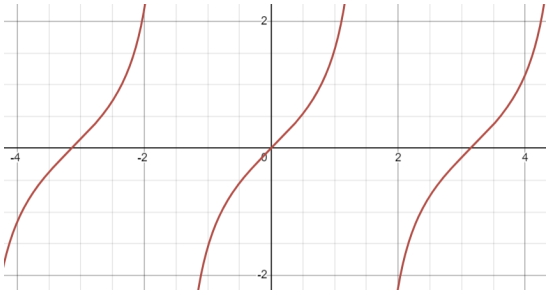


Figure 7: $y = \tan(x)$

2 Exercises
練習

Try to finish the following questions using our R Shiny Tools (https://www.math.cuhk.edu.hk/app/mathmodel/tool.html) or any other IT tools you know (e.g. Desmos, Wolfram Alpha, Microsoft Excel).
嘗試使用我們的R Shiny 工具(https://www.math.cuhk.edu.hk/app/mathmodel/tool.html) 或任何你熟悉的資訊科技工具（例如Desmos、Wolfram Alpha、Microsoft Excel）完成以下問題。

1. Modelling Population
人口建模

The World Bank dataset (https://data.worldbank.org/country) provides population data for different countries over time.
世界銀行數據集(https://data.worldbank.org/country) 提供了不同國家隨時間變化的人口數據。

(a) Consider the population of a few countries of your choice (e.g. China, USA, the United Kingdom) from 1970 to 2020, and decide what model would be best for approximating the population for each of them.
考慮你所選的幾個國家（例如中國、美國、英國）在1970 年至2020 年期間的人口數據，並決定哪種模型最適合近似每個國家的人口變化。

(b) Interpret different parameters in the model. How are they related to the data?
解釋模型中的不同參數。它們與數據有何關係？

(c) Use the model to predict the population in a future year (e.g. 2030).
使用該模型預測未來某年（例如2030 年）的人口。

2. Modelling CO2 Emission
二氧化碳排放建模

The CO2 Emission dataset (https://www.kaggle.com/datasets/debajyotipodder/co2-emission-by-vehicles, you will need to download it to see the emission data) contains data on CO2 emissions, fuel consumption, and engine size for various vehicles.
二氧化碳排放數據包含各類車輛的二氧化碳排放量、燃料消耗量及引擎尺寸數據。(https://www.kaggle.com/datasets/debajyotipodder/co2-emission-by-vehicles，需下載方可查看排放數據)

(a) Plot CO2 emissions against engine size.
繪畫二氧化碳排放量與引擎尺寸的關係圖。

(b) Fit a polynomial regression model (e.g., quadratic or cubic) to the data.
將一個多項式迴歸模型（例如二次或三次）擬合到數據中。

(c) Compare the polynomial model to a linear regression model and discuss which fits the data better.
比較多項式模型與線性迴歸模型，並討論哪個更適合擬合數據。

(d) Evaluate the model's performance using metrics like R^2 or root mean square error (RMSE).
使用R^2 或均方根誤差(RMSE) 等指標評估模型的表現。

(e) Use the model to predict CO2 emissions for a vehicle with a given engine size.
使用該模型預測具有特定引擎尺寸的車輛的二氧化碳排放量。

(f) Look for other factors that may also affect CO2 emissions and try to include them in the model.
尋找可能影響二氧化碳排放量的其他因素，並嘗試將它們納入模型中。