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**Exercise (Vectors)**

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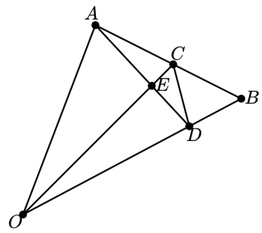
1. In the figure below, $OAB$ is a triangle. $C$ and $D$ are points on $AB$ and $OB$ respectively such that $AC : CB = 8 : 7$ and $OD : DB = 16 : 5$. $OC$ and $AD$ intersect at a point $E$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.




   (a) Express $\overrightarrow{OC}$ and $\overrightarrow{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (b) Let $\overrightarrow{OE} = r\overrightarrow{OC}$ and $\overrightarrow{AE} = k\overrightarrow{AD}$.

      i. Express $\overrightarrow{OE}$ in terms of $r$, $\mathbf{a}$ and $\mathbf{b}$.

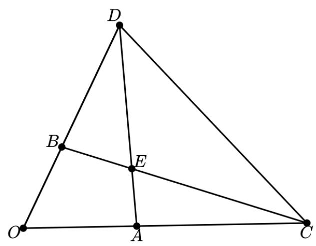
      ii. Express $\overrightarrow{OE}$ in terms of $k$, $\mathbf{a}$ and $\mathbf{b}$.
      Hence show that $r = \dfrac{6}{7}$ and $k = \dfrac{3}{5}$.

   (c) It is given that $EC : ED = 1 : 2$.

      i. Using (b), or otherwise, find $EA : EO$.

      ii. Explain why $OACD$ is a cyclic quadrilateral.

2. The figure below shows a triangle $OCD$. $A$ and $B$ are points on $OC$ and $OD$ respectively such that $OA : AC = OB : BD = 1 : h$, where $h > 0$. $AD$ and $BC$ intersect at $E$ such that $AE : ED = \mu : (1 - \mu)$ and $BE : EC = \lambda : (1 - \lambda)$, where $0 < \mu < 1$ and $0 < \lambda < 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.




   (a) By considering $\overrightarrow{OE}$, show that $\mu = \lambda$.

   (b) $F$ is a point on $CD$ such that $O$, $E$ and $F$ are collinear. Show that $OF$ is a median of $\Delta OCD$.

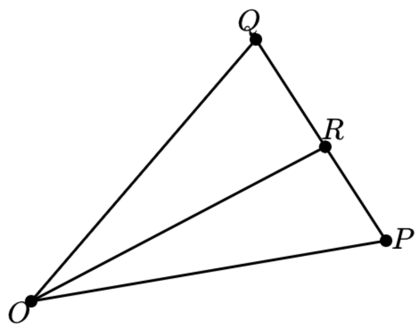
   (c) Using the above results, show that in a triangle, the centroid divides every median in $2 : 1$.

3. Given $\overrightarrow{OA} = 5\mathbf{i} - \mathbf{j}$, $\overrightarrow{OB} = -3\mathbf{i} + 5\mathbf{j}$ and $APB$ is a straight line.

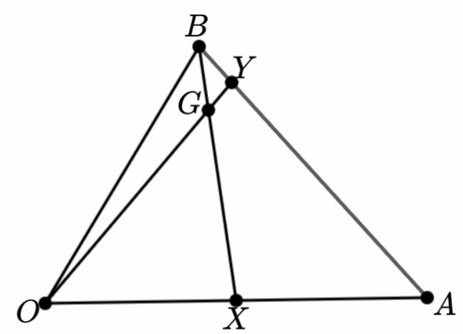
   (a) Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

   (b) If $|\overrightarrow{AP}| = 4$, find $\overrightarrow{AP}$.

4. (a) In the figure below, $OPQ$ is a triangle. $R$ is a point on $PQ$ such that $PR : RQ = r : s$. Express $\overrightarrow{OR}$ in terms of $r, s, \overrightarrow{OP}$ and $\overrightarrow{OQ}$.
   Hence show that if $\overrightarrow{OR} = m\overrightarrow{OP} + n\overrightarrow{OQ}$, then $m + n = 1$.




   (b) In the figure below, $OAB$ is a triangle. $X$ is the mid-point of $OA$ and $Y$ is a point on $AB$. $BX$ and $OY$ intersect at point $G$ where $BG : GX = 1 : 3$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.




   i. Express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

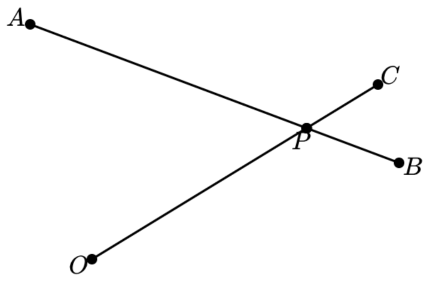
   ii. Using (a), express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (Hint: Put $\overrightarrow{OY} = k\overrightarrow{OG}$)

   iii. Moreover, $AG$ is produced to a point $Z$ on $OB$. Let $\overrightarrow{OZ} = h\overrightarrow{OB}$.

      A. Find the value of $h$.

      B. Explain whether $ZY$ is parallel to $OA$ or not.

5. In the figure below, the point $P$ divides both line segments $AB$ and $OC$ in the same ratio $3:1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.




(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Hence show that $OA$ is parallel to $BC$.

**Solutions**

1. (a) $\overrightarrow{OC} = \dfrac{7}{15}\mathbf{a} + \dfrac{8}{15}\mathbf{b}$, $\overrightarrow{AD} = -\mathbf{a} + \dfrac{16}{21}\mathbf{b}$.

   (b) i. $\overrightarrow{OE} = \dfrac{7r}{15}\mathbf{a} + \dfrac{8r}{15}\mathbf{b}$

   ii. $\overrightarrow{OE} = \mathbf{a} + k(-\mathbf{a} + \dfrac{16}{21}\mathbf{b}) = (1-k)\mathbf{a} + \dfrac{16k}{21}\mathbf{b}$

   $$\begin{cases} \dfrac{7r}{15} = 1 - k \\ \dfrac{8r}{15} = \dfrac{16k}{21} \end{cases}$$

   By solving the equations, $r = \dfrac{6}{7}$ and $k = \dfrac{3}{5}$.

   (c) i. $2 : 1 = ED : EC = \dfrac{2}{5}AD : \dfrac{1}{7}OC$, so $EA : EO = \dfrac{3}{5}AD : \dfrac{6}{7}OC = \dfrac{1}{4}(ED : EC) = 1 : 2$.

   ii. $EC : ED = EA : EO$, so $EC \cdot EO = EA \cdot ED$, which implies $OACD$ is a cyclic quadrilateral.

2. (a) $\overrightarrow{OE} = \lambda(1+h)\mathbf{a} + (1-\lambda)\mathbf{b} = (1-\mu)\mathbf{a} + \mu(1+h)\mathbf{b}$, hence $\lambda = \mu = \dfrac{1}{h+2}$.

   (b) Suppose $\overrightarrow{OF} = t\overrightarrow{OE}$, then $t\lambda(h+1) + t(1-\lambda) = 1 + h$, $t = \dfrac{h+2}{2}$, $\overrightarrow{OG} = \dfrac{1+h}{2}(\mathbf{a}+\mathbf{b}) = \dfrac{\overrightarrow{OC} + \overrightarrow{OD}}{2}$, so $F$ is the middle point of $CD$.

   (c) Let $h = 1$, $\lambda = \dfrac{1}{h+2} = 1/3$, then $AE : ED = BE : EC = 1 : 2$.

3. (a) $\overrightarrow{AB} = -8\mathbf{i} + 6\mathbf{j}$, $\left|\overrightarrow{AB}\right| = 10$.

   (b) $\overrightarrow{AP} = \pm\dfrac{2}{5}\overrightarrow{AB} = \pm(-\dfrac{16}{5}\mathbf{i} + \dfrac{12}{5}\mathbf{j})$

4. (a) $\overrightarrow{OR} = \dfrac{s}{r+s}\overrightarrow{OP} + \dfrac{r}{r+s}\overrightarrow{OQ}$, where $\dfrac{s}{r+s} + \dfrac{r}{r+s} = 1$.

   (b) i. $\overrightarrow{OG} = \dfrac{1}{4}\overrightarrow{OX} + \dfrac{3}{4}\overrightarrow{OB} = \dfrac{1}{8}\mathbf{a} + \dfrac{3}{4}\mathbf{b}$

   ii. $\overrightarrow{OY} = k\overrightarrow{OG} = \dfrac{k}{8}\mathbf{a} + \dfrac{3k}{4}\mathbf{b}$, by (a), $\dfrac{k}{8} + \dfrac{3k}{4} = 1$, $k = \dfrac{8}{7}$, $\overrightarrow{OY} = \dfrac{1}{7}\mathbf{a} + \dfrac{6}{7}\mathbf{b}$.

   iii. A. $\overrightarrow{AG} = -\dfrac{7}{8}\mathbf{a} + \dfrac{3}{4}\mathbf{b}$, suppose $\overrightarrow{AZ} = t\overrightarrow{AG}$, then $-\dfrac{7t}{8}\mathbf{a} + \dfrac{3t}{4}\mathbf{b} + \mathbf{a} = h\mathbf{b}$, $t = \dfrac{8}{7}$, $h = \dfrac{6}{7}$.

   B. $ZY = \dfrac{1}{7}\mathbf{a}$, hence $ZY$ is parallel to $OA$.

5. (a) $\overrightarrow{OP} = \dfrac{1}{4}\mathbf{a} + \dfrac{3}{4}\mathbf{b}$

   (b) $\overrightarrow{OC} = \dfrac{4}{3}\overrightarrow{OP} = \dfrac{1}{3}\mathbf{a} + \mathbf{b}$

   $\overrightarrow{BC} = \dfrac{1}{3}\mathbf{a}$, hence $OA$ is parallel to $BC$.