THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

Mathematical Modelling Project Team
mathmodel@math.cuhk.edu.hk

Exercise (Vectors and Matrices)

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Part A: Key Concepts

1. **Definition** (Vector). An object of the form $(a_1, a_2, \cdots, a_n)$ is called an **(n-dimensional) vector**. The elements $a_1, a_2, \cdots, a_n$ are called the **entries**, or **components** of the vector.
   Example. The set of all n-dimensional vectors such that $a_1, a_2, \cdots, a_n$ are real numbers is denoted by $\mathbb{R}^n$. e.g.$(3, -2, 0)$ and $(-1, 1, 4)$ are both in $\mathbb{R}^3$.
   Remark. Vectors in $\mathbb{R}^n$ can be written as column vectors
   $$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$
   rather than as **row vectors** $(a_1, a_2, \cdots, a_n)$.

2. **Definition** (Matrix). An $m \times n$ **matrix** with entries from $\mathbb{R}$ is a rectangular array of the form
   $$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$
   where each entry a_{ij} $(1 \le i \le m, 1 \le j \le n)$ is a real number. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \cdots, a_{in}$ compose the ***i*th row** of the matrix, and the entries $a_{1j}.a_{2j}, \cdots, a_{mj}$ compose the ***j*th column** of the matrix.
   Remark. In this note, we denote matrices with capital letters (e.g., A, B, and C), and we denote the entry of a matrix A that lies in row i and column j by A_{ij}. In addition, if the number of rows and columns of a matrix are equal, the matrix is called **square**.

3. **Definition** (Vector addition, Scalar multiplication).
   $\mathbb{R}^n$ is a set with the operations of coordinate-wise addition and scalar multiplication; that is, if $u = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$, $v = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$, and $c \in \mathbb{R}$, then
   $$u + v = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n) \quad \text{and} \quad cu = (ca_1, ca_2, \cdots, ca_n).$$
   Example. In $\mathbb{R}^3$, we have
   $$(3, -2, 0) + (-1, 1, 4) = (2, -1, 4) \quad \text{and} \quad -5(1, -2, 0) = (-5, 10, 0).$$

4. **Definition** (Matrix addition, Scalar multiplication).
   The set of all $m \times n$ matrices with entries from $\mathbb{R}$ is denoted by $M_{m\times n}(\mathbb{R})$, with the following operations of **matrix addition** and **scalar multiplication**: For $A, B \in M_{m\times n}(\mathbb{R})$ and $c \in \mathbb{R}$,
   $$(A + B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example.

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and

$$-3 \begin{pmatrix} 1 & 0 & 2 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & -6 \\ 9 & -6 & -9 \end{pmatrix}$$

in $M_{2\times 3}(\mathbb{R})$

5. **Definition** (Matrix Multiplication). Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the **product** of A and B, denoted AB, to be the $m \times p$ matrix such that

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj} = \sum_{k=1}^{n} A_{ik}B_{kj} \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq p$$

Remark. In order for the product AB to exist, there are restrictions regarding the relative sizes of A and B. The following mnemonic device is helpful: $(m \times n) \cdot (n \times p) = (m \times p)$. That is, the two "inner" dimensions must be equal, and the two "outer" dimensions yield the size of the product.

Example. We have

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 2 + 1 \cdot 5 \\ 0 \cdot 4 + 4 \cdot 2 + (-1) \cdot 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}$$

Note that the symbolic relationship $(2 \times 3) \cdot (3 \times 1) = (2 \times 1)$.

Definition (Identity matrix). The $n \times n$ **identity matrix** I_n is defined by $(I_n)_{ij} = 1$ if $i = j$ and $(I_n)_{ij} = 0$ if $i \neq j$. Verify that $AI_n = I_nA$ for all $n \times n$ matrices A.

6. **Definition** (Transpose). The **transpose** A^T of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^T)_{ij} = A_{ji}$.

Example.

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}.$$

7. **Definition** (Trace). The **trace** of a square matrix A is defined as the sum of all its diagonal entries and is denoted by $tr(A)$.

Example.

$$tr(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}) = 1 + 4 = 5.$$

8. **Definition** (Determinant). If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix with entries from $\mathbb{R}$, then we define the **determinant** of A, denoted $\det(A)$ or $|A|$, to be $ad - bc$.

Example. For the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

in $M_{2\times 2}(\mathbb{R})$, we have

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2 \quad \text{and} \quad \det(B) = 3 \cdot 4 - 2 \cdot 6 = 0$$

You can check that generally, $\det(A+B) \neq \det(A) + \det(B)$.

Before we extend the definition of the determinant to $n \times n$ matrices for $n \geq 3$, it is convenient to introduce the following:

Notation Given $A \in M_{n\times n}(\mathbb{R})$, for $n \geq 2$, denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$. Thus for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}),$$

we have

$$\tilde{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}, \quad \tilde{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad \text{and} \quad \tilde{A}_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$

and for

$$B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix} \in M_{4\times 4}(\mathbb{R}),$$

we have

$$\tilde{B}_{23} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -5 & 8 \\ -2 & 6 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{B}_{42} = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix}.$$

9. **Definition** (Determinant). Let $A \in M_{n\times n}(\mathbb{R})$. If $n = 1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}).$$

The scalar $\det(A)$ is called the **determinant** of A and is also denoted by $|A|$.

The scalar

$$c_{ij} = (-1)^{i+j} \det(\tilde{A}_{ij})$$

is called the **cofactor** of the entry of A in row i, column j.

Now we can express the formula for the determinant of A as

$$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \cdots + A_{1n}c_{1n}.$$

This formula is called **cofactor expansion along the first row** of A. You can check that for 2×2 matrices, this definition of the determinant of A agrees with the one above.

Example. Let

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

Using cofactor expansion along the first row of A, we obtain

$$\begin{aligned} \det(A) &= (-1)^{1+1}A_{11} \cdot \det(\tilde{A}_{11}) + (-1)^{1+2}A_{12} \cdot \det(\tilde{A}_{12}) + (-1)^{1+3}A_{13} \cdot \det(\tilde{A}_{13}) \\ &= (-1)^2(1) \cdot \det \begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3(3) \cdot \det \begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + (-1)^4(-3) \cdot \det \begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} \\ &= 1(22) - 3(26) - 3(-32) \\ &= 40. \end{aligned}$$

10. **Definition** (Non-singular). If a square matrix has a non-zero determinant, it is called a **non-singular** matrix.
Remark. The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A \in M_{n\times n}(\mathbb{R})$, then for any integer $i\,(1 \le i \le n)$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

11. **Definition** (Inverse). Let A be an $n \times n$ matrix. Then A is **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I$.
If A is invertible, then the matrix B such that $AB = BA = I$ is unique. (Try to prove this.) The matrix B is called the **inverse** of A and is denoted by A^{-1}.
Example. Verify that the inverse of

$$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}.$$

Part B: Basic Exercises

1. Compute $-2u + 4v$ where $u = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$.

2. Calculate

   (a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix}$

   (c) $4\begin{pmatrix} 2 & 5 & 3 \\ 1 & 0 & 7 \end{pmatrix}$

3. Calculate

   (a) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^2$.

   (b) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^{-1}$

4. (a) Calculate $\begin{pmatrix} 3 & 7 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}$.

   (b) Calculate the inverse of $\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$.

5. $M = \begin{pmatrix} 7 & u \\ 2 & 3 \end{pmatrix}$ and $|M| = 1$. Find the value of u.

6. $A = \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$. Calculate $A^2 - 4A$.

7. Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices. Show that $A(B + C) = AB + AC$.

Part C: Advanced Exercises

8. Let n be a positive integer.

   (a) Define $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

   i. Calculate A^2, A^3, A^4.

   ii. Can you guess what the general form of A^n is? Prove it using mathematical induction.

   iii. Similarly, find $(A^{-1})^n$.

   (b) i. Simplify $\sum_{k=0}^{n-1} 2^k$ (i.e., $2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}$).

   ii. Can you guess what the general form of $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n$ is? Prove it using mathematical induction.

9. (a) i. For any matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $tr(MN) = tr(NM)$.

   ii. Let A and B be 2×2 matrices such that $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Using the result in (i), prove that $tr(A) = 4$.

   (b) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $C\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for some non-zero vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and distinct scalars λ_1 and λ_2.

   i. Prove that $\begin{vmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p - \lambda_2 & q \\ r & s - \lambda_2 \end{vmatrix} = 0$.

   ii. Hence, prove that λ_1 and λ_2 are the roots of the equation $\lambda^2 - tr(C) \cdot \lambda + \det(C) = 0$.

10. Let $A = \begin{pmatrix} \alpha + \beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}$ where α and β are distinct real numbers. Let I be the 2×2 identity matrix.

    (a) Show that $A^2 = (\alpha + \beta)A - \alpha\beta I$.

    (b) Using (a), or otherwise, show that $(A - \alpha I)^2 = (\beta - \alpha)(A - \alpha I)$ and $(A - \beta I)^2 = (\alpha - \beta)(A - \beta I)$.

    (c) Let $X = s(A - \alpha I)$ and $Y = t(A - \beta I)$, where s and t are real numbers. Suppose $A = X + Y$.

    i. Find s and t in terms of α and β.

    ii. For any positive integer n, prove that $X^n = \frac{\beta^n}{\beta - \alpha}(A - \alpha I)$ and $Y^n = \frac{\alpha^n}{\alpha - \beta}(A - \beta I)$.

    iii. For any positive integer n, express A^n in the form of $pA + qI$, where p and q are real numbers.

    (Note: It is known that for any 2×2 matrices H and K, if $HK = KH = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $(H + K)^n = H^n + K^n$ for any positive integer n.)

11. Define $M = \begin{pmatrix} 7 & 3 \\ -1 & 5 \end{pmatrix}$. Let $X = \begin{pmatrix} a & 6a \\ b & c \end{pmatrix}$ be a non-zero real matrix such that $MX = XM$.

    (a) Express b and c in terms of a.

    (b) Prove that X is a non-singular matrix.

    (c) Denote the transpose of X be X^T. Express $(X^T)^{-1}$ in terms of a.

12. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Let I and O be the 3×3 identity matrix and zero matrix respectively.

   i. Show that $P^3 - 2P^2 - P + I = \mathbf{0}$.

   ii. Using the result of (i), or otherwise, find P^{-1}.

(b) i. Prove that $D = P^{-1}AP$.

   ii. Prove that D and A are non-singular.

   iii. Find $(D^{-1})^{100}$.
   Hence, or otherwise, find $(A^{-1})^{100}$.

Part D: Solutions

1. $-2u+4v=-2\begin{pmatrix}-2\\-2\\2\end{pmatrix}+4\begin{pmatrix}0\\4\\5\end{pmatrix}=\begin{pmatrix}4\\4\\-4\end{pmatrix}+\begin{pmatrix}0\\16\\20\end{pmatrix}=\begin{pmatrix}4\\20\\16\end{pmatrix}$

2. (a) $\begin{pmatrix}6 & 3 & 2\\-4 & 3 & 9\end{pmatrix}$

   (b) $\begin{pmatrix}1 & -1\\3 & -5\\3 & 8\end{pmatrix}$

   (c) $\begin{pmatrix}8 & 20 & -12\\4 & 0 & 28\end{pmatrix}$

3. (a) $\begin{pmatrix}8 & 5\\20 & 13\end{pmatrix}$

   (b) $\begin{pmatrix}\frac{3}{2} & -\frac{1}{2}\\-2 & 1\end{pmatrix}$

4. (a) $\begin{pmatrix}3 & 7\\-1 & 4\end{pmatrix}\begin{pmatrix}-2 & 1\\4 & 2\end{pmatrix}=\begin{pmatrix}(3)(-2)+(7)(4) & (3)(1)+7(2)\\(-1)(-2)+(4)(4) & (-1)(1)+(4)(2)\end{pmatrix}=\begin{pmatrix}22 & 17\\18 & 7\end{pmatrix}$

   (b) $\begin{pmatrix}5 & 3\\6 & 4\end{pmatrix}^{-1}=\dfrac{1}{(5)(4)-(3)(6)}\begin{pmatrix}4 & -3\\-6 & 5\end{pmatrix}=\dfrac{1}{2}\begin{pmatrix}4 & -3\\-6 & 5\end{pmatrix}$

5. 6

6. $\begin{pmatrix}4 & 16\\2 & 8\end{pmatrix}$

7. We have

$$\begin{aligned}
[A(B+C)]_{ij} &= \sum_{k=1}^{n} A_{ik}(B+C)_{kj}\\
&= \sum_{k=1}^{n} A_{ik}(B_{kj}+C_{kj})\\
&= \sum_{k=1}^{n}(A_{ik}B_{kj}+A_{ik}C_{kj})\\
&= \sum_{k=1}^{n} A_{ik}B_{kj}+\sum_{k=1}^{n} A_{ik}C_{kj}\\
&= (AB)_{ij}+(BC)_{ij}\\
&= (AB+BC)_{ij}
\end{aligned}$$

8. (a) i. $\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\4 & 1\end{pmatrix}$

   ii. $\begin{pmatrix}1 & 0\\n & 1\end{pmatrix}$

   iii. $\begin{pmatrix}1 & 0\\-n & 1\end{pmatrix}$ (by mathematical induction)

   (b) i. 2^n-1

ii. $\begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$ (by mathematical induction)

9. (a) i. $tr(MN) = ae + bg + cf + dh = ea + fc + gb + hd = tr(NM)$.

ii. Using the result in (i), we have

$$tr(A) = tr\left(B^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B\right)\right) = tr\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B\right) B^{-1}\right)$$

$$= tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} (BB^{-1})\right) = tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} I\right) = tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right) = 1 + 3 = 4$$

(b) i. Since $C\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for some non-zero vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, we have $(C - \lambda_1 I)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, i.e.,

$$\begin{pmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

If $\begin{vmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{vmatrix} \neq 0$, then the matrix inverse $\begin{pmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{pmatrix}^{-1}$ exists. Multiplying both sides of the above equation by $\begin{pmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{pmatrix}^{-1}$ gives $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, which contradicts the condition that the vector is a non-zero vector. Therefore, we have $\begin{vmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{vmatrix} = 0$. Similarly, $\begin{vmatrix} p - \lambda_2 & q \\ r & s - \lambda_2 \end{vmatrix} = 0$.

ii. From (a), we have

$$\begin{vmatrix} p - \lambda_1 & q \\ r & s - \lambda_1 \end{vmatrix} = (p - \lambda_1)(s - \lambda_1) - rq = 0,$$

from which we have

$$\lambda_1^2 - (p + s)\lambda_1 + ps - rq = 0.$$

Similarly,

$$\lambda_2^2 - (p + s)\lambda_2 + ps - rq = 0.$$

Also, note that $p + s = tr(C)$ and $ps - rq = |C|$. Therefore, λ_1, λ_2 are the two roots of the solution

$$\lambda^2 - tr(C) \cdot \lambda + \det(C) = 0.$$

10. (a) By direct calculation,

$$\text{LHS} = A^2 = \begin{pmatrix} \alpha + \beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha + \beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (\alpha + \beta)^2 - \alpha\beta & -\alpha\beta(\alpha + \beta) \\ \alpha + \beta & -\alpha\beta \end{pmatrix}$$

$$\text{RHS} = (\alpha+\beta)A - \alpha\beta I = (\alpha+\beta)\begin{pmatrix} \alpha + \beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - \alpha\beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (\alpha + \beta)^2 & -\alpha\beta(\alpha + \beta) \\ (\alpha + \beta) & 0 \end{pmatrix} - \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix}$$

Therefore, we have $A^2 = (\alpha + \beta)A - \alpha\beta I$.

(b) $(A - \alpha I)^2 = A^2 - 2\alpha A + \alpha^2 I = (\alpha + \beta)A - \alpha\beta I - 2\alpha A + \alpha^2 I = (\beta - \alpha)(A - \alpha I)$, similar for $(A - \beta I)^2$.

(c) i. $s = \dfrac{\beta}{\beta - \alpha}$, $t = \dfrac{\alpha}{\alpha - \beta}$.

ii. $X^n = s^n(A - \alpha I)^n = s^n(\beta - \alpha)^{n-1}(A - \alpha I) = \dfrac{\beta^n}{\beta - \alpha}(A - \alpha I)$, similar for Y^n.

iii. $XY = YX = \mathbf{0}$, $pA + qI = A^n = (X + Y)^n = X^n + Y^n$, hence $p = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, $q = \dfrac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}$.

11. (a) $b = -2a$, $c = -3a$.

(b) $|X| \neq 0$ when $a \neq 0$, $X = a \begin{pmatrix} 1 & 6 \\ -2 & -3 \end{pmatrix}$ is a zero matrix if $a = 0$, so $a \neq 0$.

(c) $a^{-1} \begin{pmatrix} -\frac{1}{3} & \frac{2}{9} \\ -\frac{2}{3} & \frac{1}{9} \end{pmatrix}$

12. (a) i. By direct calculation, we have $P^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $P^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$. Therefore, we have

$$P^3 - 2P^2 - P + I = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{0}.$$

ii. $P^{-1} = I + 2P - P^2 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) i. $P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $D = P^{-1}AP$ by calculation.

ii. $|A| = |D| = 4 \neq 0$

iii. $(D^{-1})^{100} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2^{100}} \end{pmatrix}$, hence $A^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{-100} \end{pmatrix} P^{-1} = \begin{pmatrix} 2^{-100} & 0 & 0 \\ 2^{-100} - 1 & 1 & 0 \\ 2^{-100} - 1 & 0 & 1 \end{pmatrix}$.