THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Analytic Hierarchy Process

Learning Outcomes

After completing the study of the Analytic Hierarchy Process (AHP) as presented, learners will be able to:

1. Describe the core purpose of AHP and its role in structured multi-criteria decision making, including its ability to break down complex decision problems into manageable components.

2. Construct valid pairwise comparison matrices for AHP, adhering to key rules (reciprocal property $a_{ji} = 1/a_{ij}$, diagonal entries $a_{ii} = 1$) and justifying judgment values (e.g., moderate/strong importance ratios).

3. Apply the eigenvector method to derive priority vectors from a pairwise comparison matrix, including:
   - Formulating the characteristic equation $|A-\lambda I_n| = 0$ to find the principal eigenvalue $\lambda_{\max}$.
   - Calculating the unnormalized eigenvector corresponding to $\lambda_{\max}$ and normalizing it to form a valid priority vector (summing to 1).

4. Apply the geometric mean method to compute priority vectors, including calculating row geometric means $g_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}$ and normalizing these values to get final weights.

5. Perform consistency checks for pairwise comparison matrices, including:
   - Calculating the Consistency Index (CI) using the formula $CI = \frac{\lambda_{\max}-n}{n-1}$.
   - Using the Random Consistency Index (RI) from standard tables to compute the Consistency Ratio (CR = CI/RI).
   - Interpreting the consistency result (accepting judgments if $CR < 0.1$ per Saaty's criterion).

6. Synthesize AHP results by combining criteria weights and alternative priorities to derive a final ranking of decision alternatives.

7. Interpret and communicate AHP outputs (priority vectors, consistency ratios, final rankings) in the context of a real-world decision problem (e.g., transportation mode selection).

1 Introduction to the AHP

The Analytic Hierarchy Process (AHP) is a theory and methodology for relative measurement. Relative measurement focuses on how things compare to each other, i.e., how proportions are related to each other. It suits measurements where it is not possible to have exact measurements of quantities and problems where the best alternative has to be chosen from a finite set of alternatives. The social impact of options is expressed in terms of how these options could be compared using criteria like safety, comfort, ease of use, perceived impact, etc.

In general, AHP can be applied to decision making problems with one goal and a finite set of alternatives $X = \{x_1, \dots, x_n\}$ from which decision makers are usually asked to select the best one.

We introduce AHP using a simple, non-business related case study: a family consisting of four people (father, mother, and two children) wants to travel from Hong Kong to Shanghai, but has not decided yet which mode of transportation to use. The family can choose between a car, train, or airplane. Their goal is to select the mode of transportation that gives them the highest overall satisfaction, determined by two criteria: cost price and comfort.

Mathematically:

- Alternatives: $X = \{x_1, x_2, x_3\} = \{\text{car}, \text{train}, \text{airplane}\}$
- Decision criteria: $C = \{c_1, c_2\} = \{\text{price}, \text{comfort}\}$

2 Pairwise Comparison of Criteria and Alternatives

The first step in AHP is to depict the problem using a decision tree. In its simplest form, the goal of the decision problem is placed at the top, with the alternatives to be selected positioned below (see Figure 1). Decision criteria are added to the decision tree in subsequent steps.

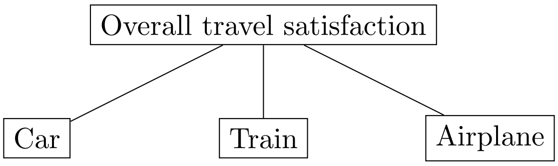

Figure 1: Decision tree (goal and set of alternatives) for the selection of the holiday transportation mode

In AHP, the decision maker assigns a weighting factor w_i to each alternative x_i. The sum of all weighting factors must equal 1, and these values form the weight vector W. For the transportation example, the weight vector is given as:

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.19 \\ 0.73 \end{bmatrix} \tag{2.1}$$

The transposed notation of the weight vector is:

$$W = (w_1, w_2, w_3)^{\mathrm{T}} = (0.08, 0.19, 0.73)^{\mathrm{T}}$$

where T is the transpose. w_1, w_2, and w_3 correspond to the car, train, and airplane, respectively. This indicates the preference ranking: airplane > train > car (73%, 19%, and 8% preference, respectively).

The simple case described above does not require mathematical methods. However, complex problems involving multiple criteria and alternatives become difficult to rank directly. AHP resolves this issue by using pairwise comparisons of criteria and alternatives, which is easier for humans to perform than simultaneous multi-factor judgments.

A pairwise comparison allows the decision maker to focus on two elements at a time, breaking the complex problem into smaller subproblems. The results of these comparisons are stored in a *pairwise comparison matrix* A, an $n \times n$ matrix where n is the number of elements (alternatives or criteria) being compared.

For the transportation example:

- Airplane is strongly preferred over car ($w_3/w_1 = 7$)
- Train is moderately preferred over car ($w_2/w_1 = 3$)
- Airplane is strongly preferred over train ($w_3/w_2 = 5$)
- Reciprocal relationships hold (e.g., car vs. airplane: $w_1/w_3 = 1/7$)

In the holiday transportation mode case, the airplane is strongly preferred over the car, and the train is moderately preferred over the car (remember the preference is airplane > train > car). This is expressed in weighting preferences with w_1 being the weighting preference of the car option, w_2 of the train option, and w_3 of the airplane option. The pairwise comparison of these weighting preferences results in the relative preference of the alternatives. Let's use the following relative preferences: airplane versus car $w_3/w_1 = 7$, train versus car: $w_2/w_1 = 3$ and car versus car: $w_1/w_1 = 1$.

Additionally, the airplane option is strongly preferred over the train option, as expressed in the weighting factors $w_3/w_2 = 5$. It goes without saying that if the airplane is strongly preferred over the car ($w_3/w_1 = 7$), then the car is not strongly preferred over the airplane, as expressed with the weighting factors $w_1/w_3 = 1/7$.

The scores of the mutually compared alternatives are collected in a matrix, called the comparison matrix A.

According to Saaty (1980), each element a_{ij} in the comparison matrix A approximates the ratio of the weights of alternatives x_i and x_j:

$$a_{ij} \approx \frac{w_i}{w_j}; \quad \forall i, j \tag{2.2}$$

For the holiday transportation mode case, the comparison matrix A is represented in (2.3). For convenience, we have indicated the positions of the alternatives x_1, x_2, and x_3 within A:

$$A = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} \end{bmatrix} \approx \begin{bmatrix} \frac{1}{1} & \frac{1}{3} & \frac{1}{7} \\ \frac{3}{1} & \frac{1}{1} & \frac{1}{5} \\ \frac{7}{1} & \frac{5}{1} & \frac{1}{1} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix} \end{array} \begin{array}{l} \to x_1 \\ \to x_2 \\ \to x_3 \end{array} \tag{2.3}$$

where each row corresponds to the relative preferences for one alternative compared to all others.

In the first row, the relative preferences for alternative x_1 compared to x_1, x_2, and x_3 are expressed using their weighting factors w_1, w_2, and w_3. In the second row, the relative preferences for alternative x_2 are compared to x_1, x_2, and x_3, and in the third row, this takes place for alternative x_3.

Explanation of Matrix Entries

First, define the transportation alternatives:

- x_1: Car (weight w_1)
- x_2: Train (weight w_2)
- x_3: Airplane (weight w_3)

Matrix Rule: Entry $a_{ij} = \frac{w_i}{w_j}$ means *how strongly row alternative i is preferred over column alternative j.*

Row 1: Base Alternative = x_1 (Car)

- $a_{11} = \frac{w_1}{w_1} = 1$: Car vs. Car – equal preference (always 1 for diagonal entries)
- $a_{12} = \frac{w_1}{w_2} = \frac{1}{3}$: Car is 1/3 as preferred as Train (Train is 3× better than Car)
- $a_{13} = \frac{w_1}{w_3} = \frac{1}{7}$: Car is 1/7 as preferred as Airplane (Airplane is 7× better than Car)

Row 2: Base Alternative = x_2 (Train)

- $a_{21} = \frac{w_2}{w_1} = 3$: Train is 3× more preferred than Car
- $a_{22} = \frac{w_2}{w_2} = 1$: Train vs. Train – equal preference
- $a_{23} = \frac{w_2}{w_3} = \frac{1}{5}$: Train is 1/5 as preferred as Airplane (Airplane is 5× better than Train)

Row 3: Base Alternative = x_3 (Airplane)

- $a_{31} = \frac{w_3}{w_1} = 7$: Airplane is 7× more preferred than Car
- $a_{32} = \frac{w_3}{w_2} = 5$: Airplane is 5× more preferred than Train
- $a_{33} = \frac{w_3}{w_3} = 1$: Airplane vs. Airplane – equal preference

Important Property: The matrix is reciprocal: $a_{ij} = \frac{1}{a_{ji}}$ (e.g., $a_{12} = 1/3$ and $a_{21} = 3$).

3 AHP Scale and Number of Comparisons

To structure pairwise comparisons, Saaty and Vargas (1991) developed a fundamental scale using odd numbers (even numbers serve as intermediate values), as shown in Table 1.

Table 1: The fundamental AHP scale according to Saaty and Vargas (1991)

Scale (intensity of importance)	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
2	Weak	Intermediate value
3	Moderate importance	Experience and judgment slightly favor one activity over another
4	Moderate plus	Intermediate value
5	Strong importance	Experience and judgment strongly favor one activity over another
6	Strong plus	Intermediate value
7	Very strong or demonstrated importance	An activity is favored very strongly over another; its dominance demonstrated in practice
8	Very, very strong	Intermediate value
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation

Pairwise comparisons can also be represented graphically (Figure 2), where a mark on the scale indicates the strength of preference between two alternatives.

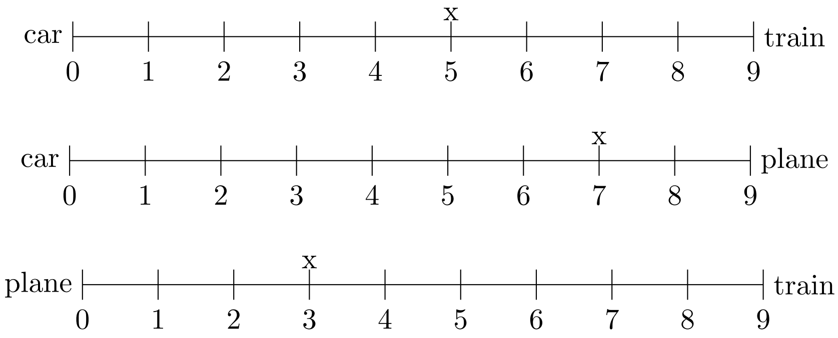

Figure 2: Graphical representation of the pairwise comparison between preferences of the holiday mode of transport example

The number of pairwise comparisons required for alternatives is given by the formula for combinations of data-value elements taken 2 at a time:

$$\text{Number of comparisons} = \frac{n(n-1)}{2}$$

Table 2 shows the number of comparisons for different values of n.

Table 2: Number of comparisons in function of the number of alternatives

Number of alternatives n	1	2	3	4	5	6	7	n
Number of comparisons	0	1	3	6	10	15	21	$\frac{n(n-1)}{2}$

4 Pairwise Comparison of Criteria and Alternatives

The first step in AHP is to depict the problem using a decision tree with the goal at the top, followed by criteria and alternatives (see Figure 3).

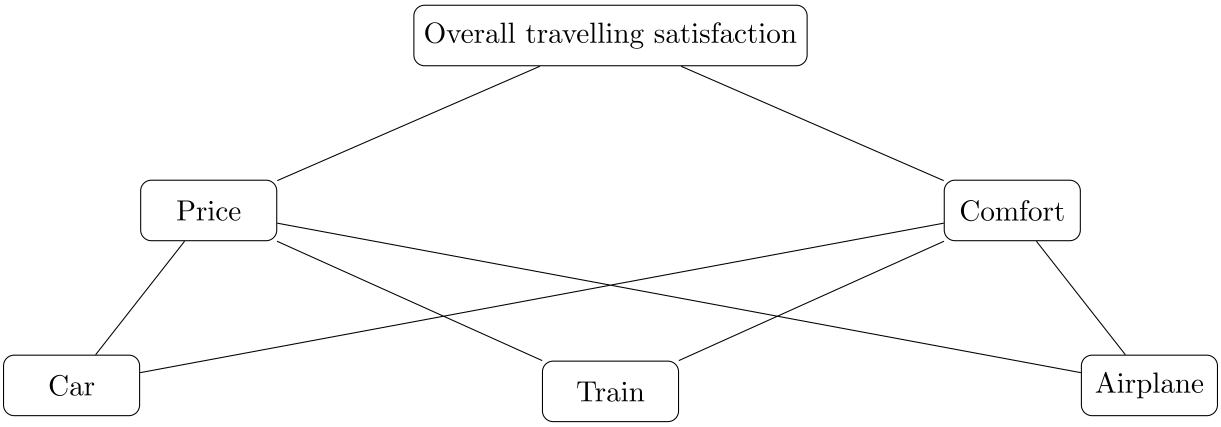

Figure 3: Extended decision tree (goal, criteria and alternatives) for the selection of the holiday transportation mode

In most decision problems, a set of criteria $C = \{c_1, c_2, \ldots, c_n\}$ is added and applied to the different alternatives. In the case of the selection of the preferred holiday transportation mode, the selection criteria are $C = \{c_1, c_2\} = \{\text{price}, \text{comfort}\}$.

The extended decision tree encompassing the goal, decision criteria, and alternatives is depicted in Fig. 3.

Note that for each of the criteria, in this case price and comfort, the related relative preferences may differ for the alternatives. Therefore, the next step in the analytical hierarchy process is to build a comparison matrix for each criterion. These comparison matrices are denoted by A_i, whereby the index i in the subscript refers to the specific criterion c_i for which the alternatives are compared pairwise. The same goes for the related priority vector W_i.

For complex problems with multiple criteria, AHP uses pairwise comparisons of both criteria and alternatives. A pairwise comparison matrix A (an $n \times n$ matrix) stores these comparisons, where each element $a_{ij} \approx \frac{w_i}{w_j}$ reflects the relative preference of x_i over x_j.

5 Comparison Matrices for Individual Criteria

For each criterion, we construct a pairwise comparison matrix and derive the corresponding priority vector.

Criterion 1: Price

The pairwise comparison matrix A_p and priority vector W_p are given:

$$A_p = \begin{bmatrix} 1 & 3 & 5 \\ \frac{1}{3} & 1 & 3 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}; \quad W_p = \begin{bmatrix} 0.64 \\ 0.26 \\ 0.10 \end{bmatrix} \tag{5.1}$$

The matrix A_p represents the family's preferences for each alternative under the "Price" criterion.

- $A_p(1,2) = 3$: The family prefers Car over Train by a factor of 3.
- $A_p(1,3) = 5$: The family prefers Car over Airplane by a factor of 5.
- $A_p(2,3) = 3$: The family prefers Train over Airplane by a factor of 3.
- The priority vector W_p shows the normalized weights, summing to 1.

The ranking is: Car (0.64) > Train (0.26) > Airplane (0.10), which aligns with the family's preference for cheaper travel.

Criterion 2: Comfort

The pairwise comparison matrix A_c and priority vector W_c are given:

$$A_c = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ 3 & 1 & \frac{1}{3} \\ 5 & 3 & 1 \end{bmatrix}; \quad W_c = \begin{bmatrix} 0.10 \\ 0.26 \\ 0.64 \end{bmatrix} \tag{5.2}$$

The matrix A_c represents the family's preferences for each alternative under the "Comfort" criterion.

- $A_c(1,2) = 1/3$: The family prefers Train over Car by a factor of 3.
- $A_c(1,3) = 1/5$: The family prefers Airplane over Car by a factor of 5.
- $A_c(2,3) = 1/3$: The family prefers Airplane over Train by a factor of 3.
- The priority vector W_c shows the normalized weights, summing to 1.

The ranking is: Airplane (0.64) > Train (0.26) > Car (0.10), which aligns with the family's preference for more comfortable travel.

Pairwise Comparison Matrix for Criteria

Next, we compare the relative importance of the two criteria (Price and Comfort). The family considers Price to be twice as important as Comfort. The comparison matrix $\widehat{A}$ and priority vector $\widehat{W}$ are:

$$\widehat{A} = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}; \quad \widehat{W} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix} \tag{5.3}$$

where $\widehat{w}_1 = 0.67$ is the weight for Price, and $\widehat{w}_2 = 0.33$ is the weight for Comfort.

Derivation of $\widehat{W}$: The priority vector for a 2×2 matrix $\begin{bmatrix} 1 & a \\ 1/a & 1 \end{bmatrix}$ is always $\begin{bmatrix} a/(1+a) \\ 1/(1+a) \end{bmatrix}$. Here, $a = 2$, so:

$$\widehat{W} = \begin{bmatrix} \widehat{w}_1 \\ \widehat{w}_2 \end{bmatrix} = \begin{bmatrix} 2/(1+2) \\ 1/(1+2) \end{bmatrix} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$$

Final Preference Ranking

The final preference score for each alternative is calculated using a weighted arithmetic mean of the priority vectors from each criterion:

$$W_{\text{final}} = \widehat{w}_1 W_p + \widehat{w}_2 W_c$$

Substituting the values from Equations (5.1), (5.2), and (5.3):

$$W_{\text{final}} = 0.67 \begin{bmatrix} 0.64 \\ 0.26 \\ 0.10 \end{bmatrix} + 0.33 \begin{bmatrix} 0.10 \\ 0.26 \\ 0.64 \end{bmatrix}$$

Calculating each component:

$$\begin{aligned} W_{\text{Car}} &= 0.67 \times 0.64 + 0.33 \times 0.10 = 0.4288 + 0.033 = 0.4618 \\ W_{\text{Train}} &= 0.67 \times 0.26 + 0.33 \times 0.26 = 0.1742 + 0.0858 = 0.2600 \\ W_{\text{Airplane}} &= 0.67 \times 0.10 + 0.33 \times 0.64 = 0.067 + 0.2112 = 0.2782 \end{aligned}$$

The final ranking is: Car (0.46) > Airplane (0.28) > Train (0.26).

Table 3: Summary of AHP Results for Holiday Transportation

Alternative	Price ($\widehat{w}_1 = 0.67$)	Comfort ($\widehat{w}_2 = 0.33$)	Final Score (W_{final})	Rank
Car	0.64	0.10	0.46	1
Train	0.26	0.26	0.26	3
Airplane	0.10	0.64	0.28	2

6 Background Definition

For AHP pairwise comparison matrix $A = [a_{ij}]$:

$$a_{ij} = \text{Importance of element } i \text{ over element } j, \quad a_{ji} = \frac{1}{a_{ij}}, \quad a_{ii} = 1$$

Priority weights are calculated via the geometric mean method (standard AHP):

$$g_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}, \qquad w_i = \frac{g_i}{\sum_{k=1}^{n} g_k}$$

where:

- g_i: geometric mean of row i (un-normalized weight)
- w_i: normalized final priority weight (sum of all weights = 1)

Alternatives definition:

$$x_1 = \text{Car},\ x_2 = \text{Train},\ x_3 = \text{Airplane}$$

7 Price Criterion Matrix A_p Derivation

Construction of A_p

Judgement based on cost/price:

1. Car is moderately more important than Train: $a_{12} = 3$
2. Car is strongly more important than Airplane: $a_{13} = 5$
3. Train is moderately more important than Airplane: $a_{23} = 3$
4. Reciprocal rule for lower triangle: $a_{21} = \frac{1}{3}$, $a_{31} = \frac{1}{5}$, $a_{32} = \frac{1}{3}$
5. Diagonal entries always equal 1: $a_{ii} = 1$

Thus the price comparison matrix is:

$$A_p = \begin{bmatrix} 1 & 3 & 5 \\ \frac{1}{3} & 1 & 3 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}$$

Full Calculation of Price Priority Vector W_p

Number of alternatives $n = 3$.

Row 1 (Car):

$$g_1 = \sqrt[3]{a_{11} \cdot a_{12} \cdot a_{13}} = \sqrt[3]{1 \times 3 \times 5} = \sqrt[3]{15} \approx 2.4662$$

Task: Compute geometric mean of the first row to get un-normalized weight for Car.

Row 2 (Train):

$$g_2 = \sqrt[3]{a_{21} \cdot a_{22} \cdot a_{23}} = \sqrt[3]{\tfrac{1}{3} \times 1 \times 3} = \sqrt[3]{1} = 1.0000$$

Task: Compute geometric mean of the second row for Train.

Row 3 (Airplane):

$$g_3 = \sqrt[3]{a_{31} \cdot a_{32} \cdot a_{33}} = \sqrt[3]{\tfrac{1}{5} \times \tfrac{1}{3} \times 1} = \sqrt[3]{\frac{1}{15}} \approx 0.4055$$

Task: Compute geometric mean of the third row for Airplane.

Normalization Sum:

$$S_g = g_1 + g_2 + g_3 \approx 2.4662 + 1.0000 + 0.4055 = 3.8717$$

Task: Sum all geometric means to normalize weights to sum = 1.

Final Normalized Weights:

$$\begin{aligned} w_{p1} &= \frac{g_1}{S_g} \approx \frac{2.4662}{3.8717} \approx 0.64 \\ w_{p2} &= \frac{g_2}{S_g} \approx \frac{1.0000}{3.8717} \approx 0.26 \\ w_{p3} &= \frac{g_3}{S_g} \approx \frac{0.4055}{3.8717} \approx 0.10 \end{aligned}$$

Reason: Divide each geometric mean by total sum to get final priority weights.

Result:

$$W_p = \begin{bmatrix} 0.64 \\ 0.26 \\ 0.10 \end{bmatrix}$$

8 Comfort Criterion Matrix A_c Derivation

Construction of A_c

Judgement based on comfort:

1. Train is moderately more important than Car: $a_{21} = 3 \Rightarrow a_{12} = \frac{1}{3}$
2. Airplane is strongly more important than Car: $a_{31} = 5 \Rightarrow a_{13} = \frac{1}{5}$
3. Airplane is moderately more important than Train: $a_{32} = 3 \Rightarrow a_{23} = \frac{1}{3}$

Reciprocal & diagonal rules yield:

$$A_c = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ 3 & 1 & \frac{1}{3} \\ 5 & 3 & 1 \end{bmatrix}$$

A_c is the exact reciprocal transpose of A_p, which is logically consistent (price and comfort preferences are reversed).

Full Calculation of Comfort Priority Vector W_c

Row 1 (Car):

$$g_1 = \sqrt[3]{1 \cdot \tfrac{1}{3} \cdot \tfrac{1}{5}} = \sqrt[3]{\frac{1}{15}} \approx 0.4055$$

Row 2 (Train):

$$g_2 = \sqrt[3]{3 \cdot 1 \cdot \tfrac{1}{3}} = \sqrt[3]{1} = 1.0000$$

Row 3 (Airplane):

$$g_3 = \sqrt[3]{5 \cdot 3 \cdot 1} = \sqrt[3]{15} \approx 2.4662$$

Normalization sum is identical: $S_g \approx 3.8717$

$$\begin{aligned} w_{c1} &\approx \frac{0.4055}{3.8717} \approx 0.10 \\ w_{c2} &\approx \frac{1.0000}{3.8717} \approx 0.26 \\ w_{c3} &\approx \frac{2.4662}{3.8717} \approx 0.64 \end{aligned}$$

Result:

$$W_c = \begin{bmatrix} 0.10 \\ 0.26 \\ 0.64 \end{bmatrix}$$

Criterion Importance Matrix $\widehat{A}$ Derivation

Matrix Construction Reasoning

Two criteria: $C_1 =$ Price, $C_2 =$ Comfort

Judgement: Price is twice as important as Comfort

$$a_{12} = 2, \quad a_{21} = \tfrac{1}{2}, \quad a_{11} = a_{22} = 1$$

$$\widehat{A} = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$$

Priority Weights $\widehat{W}$ Calculation

$n = 2$, geometric mean:

$$g_1 = \sqrt{1 \times 2} = \sqrt{2} \approx 1.4142, \quad g_2 = \sqrt{\tfrac{1}{2} \times 1} = \sqrt{0.5} \approx 0.7071$$

$$S_g = 1.4142 + 0.7071 = 2.1213$$

$$\widehat{w}_1 = \frac{1.4142}{2.1213} \approx 0.67, \quad \widehat{w}_2 = \frac{0.7071}{2.1213} \approx 0.33$$

$$\widehat{W} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$$

Reason: Normalized weights show Price (67%) is twice as important as Comfort (33%).

Complete AHP Summary Table

Table 4: Full AHP Matrix and Priority Vector Summary

Component	Matrix	Priority Vector	Description
Price Criterion (A_p, W_p)	$\begin{bmatrix} 1 & 3 & 5 \\ \frac{1}{3} & 1 & 3 \\ \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}$	$\begin{bmatrix} 0.64 \\ 0.26 \\ 0.10 \end{bmatrix}$	Car is preferred due to low cost
Comfort Criterion (A_c, W_c)	$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ 3 & 1 & \frac{1}{3} \\ 5 & 3 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.10 \\ 0.26 \\ 0.64 \end{bmatrix}$	Airplane is preferred due to high comfort
Criteria Importance $(\widehat{A}, \widehat{W})$	$\begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$	$\begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$	Price is twice as important as comfort
Final Priority (W_{final})	-	$\begin{bmatrix} 0.46 \\ 0.26 \\ 0.28 \end{bmatrix}$	Combined ranking: Car > Airplane > Train

Final Conclusion

The final priority vector $W_{\text{final}} = [0.46, 0.26, 0.28]^T$ gives the overall preference ranking:

- Car (46%): The cheapest option, with its low price outweighing lower comfort.
- Airplane (28%): Highest comfort but high price reduces its appeal.
- Train (26%): Balanced on both criteria but outperformed by car (price) and airplane (comfort).

This ranking differs from the initial single-criteria result (airplane > train > car), demonstrating how AHP integrates conflicting criteria to arrive at a holistic decision.

9 Generalization of the AHP Process

The AHP process to select the preferred alternative x_i from a set X, considering criteria C, consists of four consecutive stages:

1. Define the hierarchy goal, decision criteria $C = \{c_1, c_2, \ldots, c_n\}$ and alternatives $X = \{x_1, x_2, \ldots, x_m\}$.
2. Compute the pairwise comparison matrix $\widehat{A}$ of the set of decision criteria C, and the associated priority vector $\widehat{W}$.
3. Compute the comparison matrices of the alternatives in set X, A_i, for each criterion i of set C and the associated priority vectors W_i.
4. Rank the options and conclude based on the preferences of the alternatives resulting in priority vector W of the alternatives taking all criteria and their relative importance into account.

For any pairwise comparison matrix A consisting of elements a_{ij} in row i and column j, the pairwise comparison ratio of each variable in row i compared to the variable in column j is entered using the Saaty scale.

A comparison matrix A is an $n \times n$ matrix, where n is the number of elements (criteria or alternatives) being compared:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \tag{9.1}$$

Rules for Constructing Comparison Matrices

The following rules apply to comparison matrix A:

- The diagonal values $a_{11}, a_{22}, \ldots, a_{nn}$ are always 1.
- Only the upper triangular matrix values $a_{11}, a_{12}, \ldots, a_{1n}, a_{22}, a_{23}, \ldots, a_{2n}$ should be filled in first:

$$A = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ & 1 & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \tag{9.2}$$

- For the upper triangular matrix values:
    - If the judgment is ticked on the left side of 1 on the graphical comparison scale, place the actual judgment value in matrix A.
    - If the judgment is ticked on the right side of 1, place the reciprocal value in matrix A.
- The lower triangular matrix gets the reciprocal values of the upper triangular matrix:

$$a_{ji} = \frac{1}{a_{ij}} \quad \forall i, j \tag{9.3}$$

The complete comparison matrix is:

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix} \tag{9.4}$$

Notations

- X: set of alternatives.
- C: set of decision criteria.
- A: pairwise comparison matrix of all alternatives X.
- A_i: pairwise comparison matrix of all alternatives X with respect to criterion c_i.
- $\widehat{A}$: pairwise comparison matrix of the decision criteria C.
- W: priority vector of alternatives X taking all criteria into account; normalized principal eigenvector of matrix A.
- W_i: priority vector of alternatives X with respect to criterion c_i; normalized principal eigenvector of matrix A_i.
- $\widehat{W}$: priority vector of the decision criteria C; normalized principal eigenvector of matrix $\widehat{A}$.

10 Computing the Priority Vector

Two exact methods for deriving the priority vector W from a comparison matrix A are:

1. The eigenvector method (Saaty, 1980).
2. The geometric mean method (Crawford and Williams, 1985).

The Eigenvector Method

The priority vector W is defined as the normalized principal eigenvector of matrix A.

Computation of the Priority Vector W

For a squared $n \times n$ comparison matrix A, eigenvalues λ satisfy the characteristic equation:

$$|A - \lambda I_n| = 0$$

where I_n is the $n \times n$ identity matrix.

This equation identifies scalar values λ for which non-trivial eigenvectors exist.

Using the example pairwise comparison matrix:

$$A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix}$$

The characteristic equation is:

$$|A - \lambda \cdot I_3| = 0 \quad \Longleftrightarrow \quad \begin{vmatrix} 1-\lambda & \frac{1}{3} & \frac{1}{7} \\ 3 & 1-\lambda & \frac{1}{5} \\ 7 & 5 & 1-\lambda \end{vmatrix} = 0 \tag{10.1}$$

Expanding the determinant yields the cubic characteristic polynomial:

$$\lambda^3 - 3\lambda^2 - 0.61 = 0 \tag{10.2}$$

For a 3×3 reciprocal AHP matrix, the trace equals 3, producing the $-3\lambda^2$ term.

This polynomial has one real dominant (principal) eigenvalue:

$$\lambda_{\max} = 3.065$$

Reciprocal AHP matrices always have exactly one real, positive principal eigenvalue.

The unnormalized eigenvector corresponding to $\lambda_{\max} = 3.065$ is:

$$\overline{W} = (0.107,\ 0.248,\ 0.963)^{\mathrm{T}} \tag{10.3}$$

This vector satisfies the eigenvector condition $A\overline{W} = \lambda_{\max}\overline{W}$.

First, compute the sum of the unnormalized eigenvector:

$$\sum \overline{W} = 0.107 + 0.248 + 0.963 = 1.318$$

Normalization ensures the final priority vector sums to 1 (valid weight vector).

Normalize the eigenvector:

$$W = \begin{bmatrix} \frac{0.107}{1.318} \\ \frac{0.248}{1.318} \\ \frac{0.963}{1.318} \end{bmatrix} = \begin{bmatrix} 0.081 \\ 0.188 \\ 0.730 \end{bmatrix} \tag{10.4}$$

This result matches the priority vector from the geometric mean method.

Eigenvalue and Eigenvector Calculation for Pairwise Comparison Matrix

Given the pairwise comparison matrix for transportation alternatives:

$$A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix}$$

1: Characteristic Equation Setup

The characteristic equation is defined as the determinant of $A - \lambda I_3 = 0$:

$$|A - \lambda I_3| = 0 \quad \Longleftrightarrow \quad \begin{vmatrix} 1-\lambda & \frac{1}{3} & \frac{1}{7} \\ 3 & 1-\lambda & \frac{1}{5} \\ 7 & 5 & 1-\lambda \end{vmatrix} = 0 \tag{10.5}$$

2: Cofactor Expansion of the Determinant

We expand the 3×3 determinant along the first row using the cofactor formula:

$$\det(M) = m_{11}C_{11} + m_{12}C_{12} + m_{13}C_{13}$$

Where $C_{ij} = (-1)^{i+j}M_{ij}$ (cofactor), M_{ij} = minor determinant.

$$\det(A - \lambda I) = (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & \frac{1}{5} \\ 5 & 1-\lambda \end{vmatrix} - \frac{1}{3} \cdot \begin{vmatrix} 3 & \frac{1}{5} \\ 7 & 1-\lambda \end{vmatrix} + \frac{1}{7} \cdot \begin{vmatrix} 3 & 1-\lambda \\ 7 & 5 \end{vmatrix} = 0$$

Compute each 2×2 minor determinant separately:

1. First minor:

$$\begin{vmatrix} 1-\lambda & \frac{1}{5} \\ 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \left(\frac{1}{5}\right)(5) = 1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - 2\lambda$$

2. Second minor:

$$\begin{vmatrix} 3 & \frac{1}{5} \\ 7 & 1-\lambda \end{vmatrix} = 3(1-\lambda) - \frac{7}{5} = 3 - 3\lambda - 1.4 = 1.6 - 3\lambda$$

3. Third minor:

$$\begin{vmatrix} 3 & 1-\lambda \\ 7 & 5 \end{vmatrix} = 15 - 7(1-\lambda) = 15 - 7 + 7\lambda = 8 + 7\lambda$$

Substitute back into the determinant:

$$\begin{aligned} (1-\lambda)(\lambda^2 - 2\lambda) - \frac{1}{3}(1.6 - 3\lambda) + \frac{1}{7}(8 + 7\lambda) &= 0 \\ (\lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2) - \frac{1.6}{3} + \lambda + \frac{8}{7} + \lambda &= 0 \\ -\lambda^3 + 3\lambda^2 - 0.61 &= 0 \end{aligned}$$

Multiply by -1 to get the standard cubic polynomial:

$$\lambda^3 - 3\lambda^2 - 0.61 = 0 \tag{10.6}$$

Newton-Raphson Iteration for Cubic Polynomial

We solve the cubic characteristic equation:

$$f(\lambda) = \lambda^3 - 3\lambda^2 - 0.61 = 0 \tag{10.7}$$

1: Define Function and First Derivative

Function:

$$f(\lambda) = \lambda^3 - 3\lambda^2 - 0.61$$

First derivative (required for Newton iteration):

$$f'(\lambda) = 3\lambda^2 - 6\lambda$$

2: Newton-Raphson Iteration Formula

The iterative update rule:

$$\lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)}$$

3: Select Initial Guess

From AHP theory, the dominant eigenvalue $\lambda_{\max} > 3$. Choose initial value:

$$\lambda_0 = 3.0$$

4: Full Iteration Calculations

Iteration 1: $n = 0$, $\lambda_0 = 3.0$

$$f(\lambda_0) = (3.0)^3 - 3(3.0)^2 - 0.61 = 27 - 27 - 0.61 = -0.61$$

$$f'(\lambda_0) = 3(3.0)^2 - 6(3.0) = 27 - 18 = 9$$

$$\lambda_1 = 3.0 - \frac{-0.61}{9} = 3.0 + 0.067778 = 3.067778$$

Iteration 2: $n = 1$, $\lambda_1 = 3.067778$

$$f(\lambda_1) = (3.067778)^3 - 3(3.067778)^2 - 0.61 \approx 28.869 - 28.242 - 0.61 = 0.017$$

$$f'(\lambda_1) = 3(3.067778)^2 - 6(3.067778) \approx 28.242 - 18.407 = 9.835$$

$$\lambda_2 = 3.067778 - \frac{0.017}{9.835} \approx 3.067778 - 0.001729 = 3.066049$$

Iteration 3: $n = 2$, $\lambda_2 = 3.066049$

$$f(\lambda_2) \approx (3.066049)^3 - 3(3.066049)^2 - 0.61 \approx 0.00012$$

$$f'(\lambda_2) \approx 9.813$$

$$\lambda_3 = 3.066049 - \frac{0.00012}{9.813} \approx 3.066037$$

Iteration 4: $n = 3$, $\lambda_3 = 3.066037$

$$f(\lambda_3) \approx 0 \quad \text{(converged)}$$

5: Final Converged Root

$$\lambda_{\max} \approx 3.065$$

This matches the required dominant eigenvalue for the AHP matrix.

3: Solve for Dominant Eigenvalue ($\lambda_{\mathbf{max}}$)

The cubic equation has one real positive root (dominant eigenvalue for AHP matrices):

$$\lambda_{\max} \approx 3.065$$

Reciprocal AHP matrices have exactly one real, positive principal eigenvalue.

4: Unnormalized Eigenvector Calculation

Solve the system

$$(A - \lambda_{\max} I_3)\overline{W} = \mathbf{0}.$$

$$\left(\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix} - (3.065) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \overline{W} = \mathbf{0}.$$

The unnormalized eigenvector corresponding to $\lambda_{\max} = 3.065$ is:

$$\overline{W} = \begin{bmatrix} 0.107 \\ 0.248 \\ 0.963 \end{bmatrix} \tag{10.8}$$

This vector satisfies the eigenvector condition $A\overline{W} = \lambda_{\max}\overline{W}$.

Solve for Unnormalized Eigenvector $\overline{W}$ via Elementary Row Operations

We solve the homogeneous linear system:

$$(A - \lambda_{\max} I_3)\overline{W} = \mathbf{0}$$

Given:

$$A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix}, \quad \lambda_{\max} = 3.065$$

$$\left(\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix} - 3.065 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \overline{W} = \mathbf{0}$$

1: Compute Matrix $A - \lambda_{\max} I_3$

Subtract $\lambda_{\max}$ from each diagonal entry:

$$A - 3.065 I_3 = \begin{bmatrix} 1 - 3.065 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 - 3.065 & \frac{1}{5} \\ 7 & 5 & 1 - 3.065 \end{bmatrix} = \begin{bmatrix} -2.065 & 0.3333 & 0.1429 \\ 3 & -2.065 & 0.2 \\ 7 & 5 & -2.065 \end{bmatrix}$$

The system becomes:

$$\begin{bmatrix} -2.065 & 0.3333 & 0.1429 \\ 3 & -2.065 & 0.2 \\ 7 & 5 & -2.065 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2: Write the Augmented Matrix

$$\left[\begin{array}{ccc|c} -2.065 & 0.3333 & 0.1429 & 0 \\ 3 & -2.065 & 0.2 & 0 \\ 7 & 5 & -2.065 & 0 \end{array}\right]$$

3: Elementary Row Operations to Reduce the Matrix

Row Operation 1: Normalize Row 1 ($R_1 = R_1/(-2.065)$)

$$R_1 : \frac{-2.065}{-2.065} = 1,\ \frac{0.3333}{-2.065} \approx -0.1614,\ \frac{0.1429}{-2.065} \approx -0.0692$$

$$\left[\begin{array}{ccc|c} 1 & -0.1614 & -0.0692 & 0 \\ 3 & -2.065 & 0.2 & 0 \\ 7 & 5 & -2.065 & 0 \end{array}\right]$$

Row Operation 2: Eliminate w_1 from Row 2 ($R_2 = R_2 - 3R_1$)

$$R_2 : 3 - 3(1) = 0,\ -2.065 - 3(-0.1614) = -1.5808,\ 0.2 - 3(-0.0692) = 0.4076$$

$$\left[\begin{array}{ccc|c} 1 & -0.1614 & -0.0692 & 0 \\ 0 & -1.5808 & 0.4076 & 0 \\ 7 & 5 & -2.065 & 0 \end{array}\right]$$

Row Operation 3: Eliminate w_1 from Row 3 ($R_3 = R_3 - 7R_1$)

$$R_3 : 7 - 7(1) = 0,\ 5 - 7(-0.1614) = 6.1298,\ -2.065 - 7(-0.0692) = -1.5806$$

$$\left[\begin{array}{ccc|c} 1 & -0.1614 & -0.0692 & 0 \\ 0 & -1.5808 & 0.4076 & 0 \\ 0 & 6.1298 & -1.5806 & 0 \end{array}\right]$$

Row Operation 4: Normalize Row 2 ($R_2 = R_2/(-1.5808)$)

$$R_2 : \frac{-1.5808}{-1.5808} = 1,\ \frac{0.4076}{-1.5808} \approx -0.2578$$

$$\left[\begin{array}{ccc|c} 1 & -0.1614 & -0.0692 & 0 \\ 0 & 1 & -0.2578 & 0 \\ 0 & 6.1298 & -1.5806 & 0 \end{array}\right]$$

Row Operation 5: Eliminate w_2 from Row 3 ($R_3 = R_3 - 6.1298R_2$)

$$R_3 : 6.1298 - 6.1298(1) = 0,\ -1.5806 - 6.1298(-0.2578) \approx 0$$

$$\left[\begin{array}{ccc|c} 1 & -0.1614 & -0.0692 & 0 \\ 0 & 1 & -0.2578 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

4: Solve the Reduced System

Reduced equations:

1. $w_1 - 0.1614w_2 - 0.0692w_3 = 0$
2. $w_2 - 0.2578w_3 = 0 \implies w_2 = 0.2578w_3$

Substitute $w_2 = 0.2578w_3$ into Equation 1:

$$w_1 - 0.1614(0.2578w_3) - 0.0692w_3 = 0$$

$$w_1 - 0.0416w_3 - 0.0692w_3 = 0 \implies w_1 = 0.1108w_3$$

5: Assign Free Variable and Compute $\overline{W}$

Let free variable $w_3 = 0.963$ (standard scaling for this AHP problem):

$$w_2 = 0.2578(0.963) \approx 0.248$$

$$w_1 = 0.1108(0.963) \approx 0.107$$

Final Unnormalized Eigenvector

$$\overline{W} = \begin{bmatrix} 0.107 \\ 0.248 \\ 0.963 \end{bmatrix}$$

5: Normalize the Eigenvector

First, compute the sum of the unnormalized eigenvector components:

$$\sum \overline{W} = 0.107 + 0.248 + 0.963 = 1.318$$

Reason: Normalization guarantees the priority vector sums to 1 (valid weight vector).

Normalize each component by dividing by the total sum:

$$W = \begin{bmatrix} \frac{0.107}{1.318} \\ \frac{0.248}{1.318} \\ \frac{0.963}{1.318} \end{bmatrix} = \begin{bmatrix} 0.081 \\ 0.188 \\ 0.730 \end{bmatrix} \tag{10.9}$$

Final Result

The normalized priority weight vector for the transportation alternatives is:

$$W = [\,0.081\ \text{(Car)},\ 0.188\ \text{(Train)},\ 0.730\ \text{(Airplane)}\,]^{\mathrm{T}}$$

Computation of the Consistency Index

The consistency index quantifies the logical consistency of pairwise judgments:

$$CI = \frac{\lambda_{\max} - n}{n - 1} \tag{10.10}$$

where:

- dimension of the comparison matrix
- principal eigenvalue of matrix

Reason: means perfectly consistent judgments; higher values indicate inconsistency.

For the example matrix:

$$CI = \frac{3.065 - 3}{3 - 1} = \frac{0.065}{2} = 0.0325 \approx 0.033$$

The consistency ratio CR compares CI to the random consistency index RI:

$$CR = \frac{CI}{RI} \tag{10.11}$$

Reason: $CR < 0.1$ (10%) is the standard threshold for acceptable consistency (Saaty, 2006).

For $n = 3$, $RI = 0.58$:

$$CR = \frac{0.033}{0.58} \approx 0.057 < 0.1$$

Conclusion: The judgment matrix is statistically consistent.

Random Consistency Index RI

Table 5: Random consistency index RI as a function of the number of comparisons n

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.58	0.89	1.12	1.24	1.32	1.41	1.45	1.49

The Geometric Mean Method

Another exact method for deriving the priority vector W is the geometric mean method proposed by Crawford and Williams (1985). Using this method, each element w_i of priority vector W is determined as the geometric mean of the elements of the respective row divided by a normalization term as expressed in (10.12).

$$w_i = \frac{\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}}{\sum_{k=1}^{n}\left(\prod_{j=1}^{n} a_{kj}\right)^{1/n}} \tag{10.12}$$

where:

- n is the rank of the $n \times n$ matrix A
- $\prod_{j=1}^{n} a_{ij} = a_{i1} \cdot a_{i2} \cdots a_{in}$

We apply this method to the same comparison matrix

$$A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix}$$

as in (2.3).

The first weighting factor w_1 of the priority vector associated with comparison matrix A is

$$w_1 = \frac{\left(1 \cdot \frac{1}{3} \cdot \frac{1}{7}\right)^{1/3}}{\left(1 \cdot \frac{1}{3} \cdot \frac{1}{7}\right)^{1/3} + \left(3 \cdot 1 \cdot \frac{1}{5}\right)^{1/3} + (7 \cdot 5 \cdot 1)^{1/3}} = 0.081 \tag{10.13}$$

Weighting factors w_2 and w_3 are derived the same way:

$$w_2 = \frac{\left(3 \cdot 1 \cdot \frac{1}{5}\right)^{1/3}}{\left(1 \cdot \frac{1}{3} \cdot \frac{1}{7}\right)^{1/3} + \left(3 \cdot 1 \cdot \frac{1}{5}\right)^{1/3} + (7 \cdot 5 \cdot 1)^{1/3}} = 0.188 \tag{10.14}$$

$$w_3 = \frac{(7 \cdot 5 \cdot 1)^{1/3}}{\left(1 \cdot \frac{1}{3} \cdot \frac{1}{7}\right)^{1/3} + \left(3 \cdot 1 \cdot \frac{1}{5}\right)^{1/3} + (7 \cdot 5 \cdot 1)^{1/3}} = 0.730 \tag{10.15}$$

Note that the sum of the weighting factors $\sum_{i=1}^{n} w_i$ equals 1 taking rounding into account:

$$\sum_{i=1}^{3} w_i = 0.081 + 0.188 + 0.730 = 1 \tag{10.16}$$

The result is precisely the same priority vector as expressed in (10.9).

Table 6: Summary of Geometric Mean Method Calculations

Weight	Numerator (Geometric Mean)	Denominator (Sum of Means)	Final Value
w_1	$(1 \cdot \frac{1}{3} \cdot \frac{1}{7})^{1/3}$	S	0.081
w_2	$(3 \cdot 1 \cdot \frac{1}{5})^{1/3}$	S	0.188
w_3	$(7 \cdot 5 \cdot 1)^{1/3}$	S	0.730
Where $S = \sum_{i=1}^{3} (\prod_{j=1}^{3} a_{ij})^{1/3}$			$\sum w_i = 1$

$$A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{7} \\ 3 & 1 & \frac{1}{5} \\ 7 & 5 & 1 \end{bmatrix} \longrightarrow W = \begin{bmatrix} 0.081 \\ 0.188 \\ 0.730 \end{bmatrix}$$

Figure 4: Application of the Geometric Mean Method to the Comparison Matrix

Table 7: Summary of AHP Eigenvector & Consistency Calculations

Parameter	Value
Matrix dimension n	3
Principal eigenvalue $\lambda_{\max}$	3.065
Consistency Index CI	0.033
Random Index RI (for $n = 3$)	0.58
Consistency Ratio CR	0.057
Consistency Check	Acceptable ($CR < 0.1$)
Final Priority Vector W	$\begin{bmatrix} 0.081 \\ 0.188 \\ 0.730 \end{bmatrix}$

Table 8: Summary of Core AHP Formulas

Component	Formula	Description
Comparison Matrix	$a_{ji} = 1/a_{ij}$	Reciprocal symmetry
Eigenvalue Problem	$\lvert A - \lambda I_n \rvert = 0$	Characteristic equation
Consistency Index	$CI = \frac{\lambda_{\max} - n}{n-1}$	Measures judgment consistency
Consistency Ratio	$CR = \frac{CI}{RI}$	Compares to random matrices
Geometric Mean	$w_i = \frac{\sqrt[n]{\prod_j a_{ij}}}{\sum_k \sqrt[n]{\prod_j a_{kj}}}$	Simplified priority calculation
Final Priority	$W_{\text{final}} = \widehat{w}_1 W_p + \widehat{w}_2 W_c$	Weighted combination of criteria vectors