THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercises on Eigenvalues and Eigenvectors of a Matrix

Example 1. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

The characteristic polynomial of A is

$$f(\lambda) = \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda & -1 \\ -4 & 4 & \lambda - 5 \end{vmatrix}.$$

We compute the determinant using co-factor expansion along the first row:

$$\begin{aligned}
\det(\lambda I - A) &= (\lambda - 1) \cdot \det \begin{pmatrix} \lambda & -1 \\ 4 & \lambda - 5 \end{pmatrix} - (-2) \cdot \det \begin{pmatrix} -1 & -1 \\ -4 & \lambda - 5 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} -1 & \lambda \\ -4 & 4 \end{pmatrix} \\
&= (\lambda - 1)\left[\lambda(\lambda - 5) - (-1)(4)\right] + 2\left[(-1)(\lambda - 5) - (-1)(-4)\right] + 1\left[(-1)(4) - \lambda(-4)\right] \\
&= (\lambda - 1)(\lambda^2 - 5\lambda + 4) + 2(-\lambda + 5 - 4) + (-4 + 4\lambda) \\
&= (\lambda - 1)(\lambda - 1)(\lambda - 4) + 2(-\lambda + 1) + 4\lambda - 4 \\
&= (\lambda^2 - 2\lambda + 1)(\lambda - 4) - 2\lambda + 2 + 4\lambda - 4 \\
&= \lambda^3 - 4\lambda^2 - 2\lambda^2 + 8\lambda + \lambda - 4 + 2\lambda - 2 \\
&= \lambda^3 - 6\lambda^2 + 11\lambda - 6.
\end{aligned}$$

Thus, the characteristic polynomial is $f(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$.

Example 2. Consider the matrix of Example 1. The characteristic polynomial is

$$f(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6.$$

The possible integer roots of $f(\lambda)$ are $\pm 1, \pm 2, \pm 3$ and ± 6. By substituting these values in $f(\lambda)$, we find $f(1) = 0$, so $\lambda = 1$ is a root of $f(\lambda)$. Hence $(\lambda - 1)$ is a factor of $f(\lambda)$. Dividing $f(\lambda)$ by $(\lambda - 1)$, we obtain

$$f(\lambda) = (\lambda - 1)(\lambda^2 - 5\lambda + 6).$$

Factoring $\lambda^2 - 5\lambda + 6$, we have

$$f(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

The eigenvalues of A are then

$$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3.$$

To find an eigenvector $\mathbf{x}_1$ associated with $\lambda_1 = 1$, we form the linear system

$$(1 I_3 - A)\mathbf{x} = \mathbf{0},$$

$$\begin{bmatrix} 1 - 1 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & 1 - 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A solution is

$$\begin{bmatrix} -\frac{1}{2}r \\ \frac{1}{2}r \\ r \end{bmatrix}$$

for any real number r. Thus for $r = 2$,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_1 = 1$.

To find an eigenvector associated with $\lambda_2 = 2$, we form the linear system

$$(2I_3 - A)\mathbf{x} = \mathbf{0},$$

$$\begin{bmatrix} 2-1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A solution is

$$\begin{bmatrix} -\frac{1}{2}r \\ \frac{1}{4}r \\ r \end{bmatrix}$$

for any real number r. Thus for $r = 4$,

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_2 = 2$.

To find an eigenvector associated with $\lambda_3 = 3$, we form the linear system

$$(3I_3 - A)\mathbf{x} = \mathbf{0},$$

and find that a solution is

$$\begin{bmatrix} -\frac{1}{4}r \\ \frac{1}{4}r \\ r \end{bmatrix}$$

for any real number r. Thus for $r = 4$,

$$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_3 = 3$.

Summary Table

Eigenvalue	Corresponding Eigenvector
$\lambda_1 = 1$	$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$
$\lambda_2 = 2$	$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$
$\lambda_3 = 3$	$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

Example 3. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution

Step 1: Find the eigenvalues of A. The characteristic equation turns out to involve a cubic polynomial that can be factored:

$$0 = \det(\lambda I_3 - A) = \lambda^3 + 3\lambda^2 - 4 = (\lambda - 1)(\lambda + 2)^2$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Step 2: Find three linearly independent eigenvectors of A. This is the crucial step. If it fails, A cannot be diagonalized.

1. For $\lambda = 1$: We have to solve the homogeneous system $(1I_3 - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 0 & -3 & -3 \\ 3 & 6 & 3 \\ -3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

2. For $\lambda = -2$: We have to solve the homogeneous system $(-2I_3 - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now,

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set (verify).

Step 3: Construct P from the vectors in Step 2. Using the order chosen in **Step 2** form

$$P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Step 4: Construct D from the corresponding eigenvalues. The order of the eigenvalues matches the order chosen for P. Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Let us check that P and D really work. To avoid confusion, we simply verify that $AP = PD$. This is equivalent to $A = PDP^{-1}$ when P is invertible. We compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

Summary of Diagonalization

Eigenvalue	Eigenvector	Position in P / D
$\lambda_1 = 1$	$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$	Column 1 / D_{11}
$\lambda_2 = -2$	$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$	Column 2 / D_{22}
$\lambda_3 = -2$	$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$	Column 3 / D_{33}

Diagonalization Result:

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$