THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercises on Solutions to Linear First-Order ODEs

1 First-Order Linear Equations

We learned that a first-order linear inhomogeneous ordinary differential equation (ODE) for the unknown function $x = x(t)$ has the standard form

$$\dot{x} + p(t)x = q(t). \tag{1.1}$$

(To be precise, we require $q(t)$ is not identically zero.)

A first-order linear homogeneous ODE for $x = x(t)$ has the standard form

$$\dot{x} + p(t)x = 0. \tag{1.2}$$

We call (1.2) the *associated homogeneous equation* to the inhomogeneous equation (1.1).

In (1.2) the input signal is identically zero. We call this the *null signal*. It corresponds to letting the system evolve in isolation without any external "disturbance".

- In the bank example: if there are no deposits and no withdrawals, the input is 0.

2 Solutions to the Homogeneous Equations

The homogeneous linear equation (1.2) is separable. We find its solution as follows:

- Separate variables:
$$\frac{dx}{x} = -p(t)\,dt.$$
- Integrate:
$$\ln|x| = -\int p(t)\,dt + c_1.$$
(We use c_1 to save C for later.)
- Exponentiate:
$$|x| = e^{c_1} e^{-\int p(t)\,dt}.$$
- Rename e^{c_1} as C:
$$|x| = Ce^{-\int p(t)\,dt}, \quad C > 0.$$
- Drop the absolute value and recover the lost solution $x(t) = 0$: This gives the general solution to (1.2):

$$x(t) = Ce^{-\int p(t)\,dt}, \quad C \in \mathbb{R}. \tag{2.1}$$

A useful notation is to choose one specific solution to equation (1.2) and call it $x_h(t)$. Then the solution (2.1) shows the general solution to the equation is

$$x(t) = Cx_h(t). \tag{2.2}$$

There is a subtle point here: formula (2.2) requires us to choose one solution to name x_h, but it does not matter which one we choose. We can state this somewhat awkwardly as "choose an arbitrary specific solution". A typical choice is to set the parameter $C = 1$, but this is not necessary.

Example 1

Solve $\dot{x} + 2tx = 0$.

Solution.

- Separate variables:
$$\frac{dx}{x} = -2t\,dt.$$
- Integrate:
$$\ln|x| = -\int 2t\,dt = -t^2 + c_1.$$
- Exponentiate and substitute C for e^{c_1}:
$$|x| = e^{c_1}e^{-t^2} = Ce^{-t^2}.$$
- Drop the absolute value and recover the lost solution:
$$x(t) = Ce^{-t^2}.$$

In this example, an obvious choice for x_h is $x_h(t) = e^{-t^2}$. It is clear the general solution to the example is
$$x(t) = Cx_h(t), \quad C \in \mathbb{R}.$$

3 Solution to Inhomogeneous ODEs Using Integrating Factors

We start with the integrating factors formula: the general solution to the inhomogeneous first-order linear ODE (1.1) ($\dot{x} + p(t)x = q(t)$) is
$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + C\right), \quad \text{where } u(t) = e^{\int p(t)\,dt}. \tag{3.1}$$

The function u is called an *integrating factor*.

This method, due to Euler, is easy to apply. We deduce it by the method of optimism, i.e., we introduce an integrating factor u and hope that it will help us.

Proof: We start with the product rule for differentiation
$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$$
and the equation (1.1):
$$\dot{x} + p(t)x = q(t).$$

Multiply both sides of the equation by some function $u(t)$, whose value we will determine later:

$$u\dot{x} + upx = uq. \tag{3.2}$$

In order to be able to apply the product rule we want the sum on the left-hand side of the equation to have the form $\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$. There may be many functions u for which the left-hand side has this form; we only need to find one of them. To do this, note that

$$\frac{d}{dt}(ux) = u\dot{x} + upx \quad \Leftrightarrow \quad u\dot{x} + \dot{u}x = u\dot{x} + upx \quad \Leftrightarrow \quad \dot{u} = up.$$

The last equation is a separable ODE for the unknown function u:

$$\frac{du}{u} = p(t)\,dt$$

and so:

$$\ln|u| = \int p(t)\,dt$$

$$u = e^{\int p(t)\,dt}. \tag{3.3}$$

Remember, we are looking for just one u, so any choice of anti-derivative of $p(t)$ in equation (3.3) will do.

Now replace the left-hand side of (3.2) by $\frac{d}{dt}(ux)$ and solve for x:

$$\begin{aligned}
u\dot{x} + upx &= uq \\
\frac{d}{dt}(ux) &= uq \\
u(t)x(t) &= \int u(t)q(t)\,dt + c \\
x(t) &= \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + c\right).
\end{aligned}$$

This last equation is exactly the formula (3.1) we want to prove.

Example 2

Solve the ODE $\dot{x} + 2x = e^{3t}$ using the method of integrating factors.

Solution. Until you are sure you can rederive (3.1) in every case it is worthwhile practicing the method of integrating factors on the given differential equation. (At the end, we will model a solution that just plugs into (3.1).)

Multiply both sides by u:

$$u\dot{x} + 2u(t)x(t) = u(t) \cdot e^{3t}. \tag{3.4}$$

Next, find an integrating factor u so that the left-hand side is equal to $\frac{d}{dt}(ux)$ (which equals $u\dot{x} + \dot{u}x$).

$$u\dot{x} + \dot{u}x = u\dot{x} + 2ux$$

$$\begin{aligned} \Rightarrow \quad \dot{u} &= 2u \\ u(t) &= e^{2t} \quad \text{(we choose any one } u \text{ that works).} \end{aligned}$$

Now substitute $u(t) = e^{2t}$ into (3.4), then replace the left-hand side by $\frac{d}{dt}(ux)$ and solve for x.

$$\begin{aligned} &\frac{d}{dt}(e^{2t}x) = e^{2t}e^{3t} \\ \Rightarrow \quad & e^{2t}x = \frac{1}{5}e^{5t} + C \quad \text{(integrate the previous equation)} \\ \Rightarrow \quad & x(t) = \frac{1}{5}e^{3t} + Ce^{-2t} \quad \text{(solve for } x(t)\text{).} \end{aligned}$$

Here is a model of the same solution using (3.1) directly.

- Integrating factor: $u(t) = e^{\int 2\,dt} = e^{2t}$ (choose any one possibility).
- Solution:

$$\begin{aligned} x(t) &= \frac{1}{u(t)}\left(\int u(t)e^{3t}\,dt + C\right) \\ &= e^{-2t}\int e^{5t}\,dt \\ &= e^{-2t}\left(\frac{1}{5}e^{5t} + C\right) \\ &= \frac{1}{5}e^{3t} + Ce^{-2t}. \end{aligned}$$

4 Comparing the Integrating Factor u and x_h

Recall that in section 2 we fixed one solution to the homogeneous equation (1.2) and called it x_h. The formula for x_h is

$$x_h(t) = e^{-\int p(t)\,dt},$$

where we can pick any one choice for the antiderivative. Comparing this with the formula for the integrating factor

$$u = e^{\int p(t)\,dt}$$

we get the following relationship between the two functions:

$$x_h(t) = \frac{1}{u(t)}.$$

Example 3

Solve the ODE

$$\frac{dy}{dx} - y = e^{3x}.$$

Solution.

- Identify standard form coefficients: The ODE is $\frac{dy}{dx} + p(x)y = q(x)$ with $p(x) = -1$, $q(x) = e^{3x}$. Compute the integrating factor $\mu(x) = e^{\int p(x)dx} = e^{\int -1dx} = e^{-x}$.
- Multiply both sides of the ODE by the integrating factor:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{2x}$$

  Rewrite the left-hand side as a derivative of a product:

$$\frac{d}{dx}\left(e^{-x}y\right) = e^{2x}$$

- Integrate both sides with respect to x:

$$e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

- Solve for the unknown function $y(x)$: Multiply both sides by e^x to isolate y:

$$y = e^x\left(\frac{1}{2}e^{2x} + C\right)$$

Exercises

1. Solve the ODE:

$$y' + y = xe^{-x} + 1.$$

2. Solve the initial value problem for $y(t)$ with $t > 0$:

$$\begin{cases} ty' + 2y = \dfrac{\sin t}{t} \\ y\left(\dfrac{\pi}{2}\right) = 0 \end{cases}$$

Exercises

1. Solve the ODE:

$$y' + y = xe^{-x} + 1.$$

Solution.

- Identify standard form coefficients: The ODE is $y'+p(x)y = q(x)$ with $p(x) = 1$, $q(x) = xe^{-x} + 1$. Compute the integrating factor $\mu(x) = e^{\int 1dx} = e^x$.
- Multiply both sides of the ODE by the integrating factor:

$$e^x y' + e^x y = x + e^x$$

Rewrite the left-hand side as a derivative of a product:

$$\frac{d}{dx}\left(e^x y\right) = x + e^x$$

- Integrate both sides with respect to x:

$$e^x y = \int (x + e^x)\, dx = \frac{1}{2}x^2 + e^x + C$$

- Solve for the unknown function $y(x)$: Divide both sides by e^x to isolate y:

$$y = \frac{1}{2}x^2 e^{-x} + 1 + Ce^{-x}$$

2. Solve the initial value problem for $y(t)$ with $t > 0$:

$$\begin{cases} ty' + 2y = \dfrac{\sin t}{t} \\ y\left(\dfrac{\pi}{2}\right) = 0 \end{cases}$$

Solution.

- Rewrite the ODE in standard form and compute the integrating factor: Divide by t: $y' + \frac{2}{t}y = \frac{\sin t}{t^2}$, where $p(t) = \frac{2}{t}$, $q(t) = \frac{\sin t}{t^2}$. Integrating factor: $\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$.
- Multiply both sides of the ODE by the integrating factor:

$$t^2 y' + 2ty = \sin t$$

Rewrite the left-hand side as a derivative of a product:

$$\frac{d}{dt}\left(t^2 y\right) = \sin t$$

- Integrate both sides with respect to t:

$$t^2 y = \int \sin t dt = -\cos t + C$$

- Solve for $y(t)$ and apply the initial condition: General solution: $y = \frac{-\cos t + C}{t^2}$. Substitute $t = \frac{\pi}{2}, y = 0$: $0 = \frac{0+C}{(\pi/2)^2} \implies C = 0$. Particular solution: $y = -\frac{\cos t}{t^2}$.