THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercises on Numerical Solutions of Differential Equations in `Python`

Numerical methods solve differential equations by computing discrete approximate solutions from initial conditions using small iterative steps, with accuracy/efficiency dependent on the algorithm and time step; common methods include Euler's and Runge-Kutta.

MATLAB provides ODE solvers: ode45 (general-purpose Runge-Kutta) and four stiff solvers (ode15s, ode23s, ode23t, ode2).

Stiff ODEs involve solutions with widely varying time scales (fast/slow components), causing standard methods to become unstable unless using extremely small time steps; stiffness depends on the numerical method, not just the equation.

To use MATLAB ODE solvers, users define an ODE function, initial conditions, and time span; solvers return time/solution vectors for plotting.

MATLAB's dsolve is a symbolic (analytical) solver that fails for complex, nonlinear, or non-closed-form ODEs, unlike numerical solvers.

The table below provides the direct correspondence between MATLAB ODE solvers and their equivalent Python methods using `scipy.integrate.solve_ivp` (numerical) and `sympy.dsolve` (symbolic).

MATLAB Solver	Python `solve_ivp` Method	Purpose
`ode45`	`RK45`	General-purpose (default)
`ode15s`	`Radau`	Stiff systems
`ode23s`	`BDF`	Stiff systems
`ode23t`	`LSODA`	Stiff/non-stiff switch
`ode23tb`	`Radau`	Stiff systems
`dsolve`	`sympy.dsolve`	Analytical (symbolic) solution

Example 1 Employ `Python` to plot the solution of the initial value problem:

$$y' = 4x, \quad y(0) = 1, \quad x \in [0, 2.5]$$

Table 1: Summary of Initial Value Problem (Example 1)

Item	Details
ODE	$y' = 4x$
Initial Condition	$y(0) = 1$
Interval	$x \in [0,\ 2.5]$
Analytical Solution	$y = 2x^2 + 1$

Example 2 Employ `Python` to plot the solution of the initial value problem:

$$y' + 3y = xe^{2x}, \quad y(2) = -4, \quad x \in [2, 2.7].$$

Exact Analytical Solution (from `SymPy`)

$$y(x) = \left(\frac{x}{5} - \frac{1}{25}\right) e^{2x} + \left(-4e^{6} - \frac{9}{25}e^{4}\right) e^{-3x}.$$

Simplified Form

$$y(x) = \frac{5x-1}{25}\, e^{2x} + Ce^{-3x},$$

where

$$C = -4e^{6} - \frac{9}{25}e^{4}.$$

Table 2: Summary of Initial Value Problem (Example 2)

Item	Information
ODE	$y' + 3y = xe^{2x}$
Initial Condition	$y(2) = -4$
Interval	$x \in [2,\ 2.7]$
Exact Solution	Given by `SymPy`
Simplified Form	$y(x) = \frac{5x-1}{25}e^{2x} + Ce^{-3x}$
Constant C	$-4e^{6} - \frac{9}{25}e^{4}$

Example 3 Employ `Python` to solve and plot the solution of the initial value problem:

$$\begin{cases} \dfrac{dx}{dt} = y^2 - x \\ \dfrac{dy}{dt} = 0.5x^2 - y \end{cases} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in [0, 3].$$

Table 3: Summary of Nonlinear System IVP (Example 3)

Item	Details
ODE System	$\dot{x} = y^2 - x,\ \dot{y} = 0.5x^2 - y$
Initial Condition	$x(0) = 1,\ y(0) = 0$
Time Interval	$t \in [0,\ 3]$
Problem Type	Nonlinear autonomous system
Solution Method	`Python` numerical solver

Example 4 System of First-Order ODEs with Fixed Time Step

Employ `Python` to solve and plot the solution of the initial value problem:

$$\begin{cases} \dfrac{dy_2}{dt} + 3y_2 = 6y_1 \\ \dfrac{dy_1}{dt} + 5y_1 = \sin(2t) \end{cases} \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \quad t \in [0, 0.5].$$

Rearranged into standard form:

$$\begin{cases} \dfrac{dy_1}{dt} = -5y_1 + \sin(2t) \\ \dfrac{dy_2}{dt} = 6y_1 - 3y_2 \end{cases}$$

Table 4: Summary of Example 4 ODE System

Item	Details
Original System	$y_2' + 3y_2 = 6y_1,\ y_1' + 5y_1 = \sin(2t)$
Standard Form	$y_1' = -5y_1 + \sin(2t),\ y_2' = 6y_1 - 3y_2$
Initial Condition	$y_1(0) = 5,\ y_2(0) = -2$
Time Interval	$t \in [0,\ 0.5]$
Problem Type	Linear system of ODEs

Example 5 3-Variable System of ODEs with Fixed Time Step

Employ `Python` to solve and plot the solution of the initial value problem:

$$\begin{cases} \dfrac{dx}{dt} = y - z \\ \dfrac{dy}{dt} = 0.45z - e^{-t} \\ \dfrac{dz}{dt} = -0.25xy + t^2 \end{cases} \qquad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 7 \end{bmatrix}, \quad t \in [0, 4].$$

Table 5: Summary of 3-Variable ODE System (Example 5)

Component	**Details**
ODE 1	$\dfrac{dx}{dt} = y - z$
ODE 2	$\dfrac{dy}{dt} = 0.45z - e^{-t}$
ODE 3	$\dfrac{dz}{dt} = -0.25xy + t^2$
Initial Condition	$x(0) = -2,\ y(0) = -5,\ z(0) = 7$
Time Interval	$t \in [0,\ 4]$
Problem Type	3-variable nonlinear ODE system

Example 6 Stiff 3-Variable ODE System (Robertson Reaction Kinetics)

Employ `Python` to solve and plot the solution of the initial value problem:

$$\begin{cases} \dfrac{dy_1}{dt} = -k_1 y_1 + k_3 y_2 y_3 \\ \dfrac{dy_2}{dt} = k_1 y_1 - k_2 y_2^2 - k_3 y_2 y_3 \\ \dfrac{dy_3}{dt} = k_2 y_2^2 \end{cases} \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

with rate constants $k_1 = 0.04$, $k_2 = 10^4$, $k_3 = 3 \times 10^7$, over the interval $t \in [0, 4 \times 10^6]$.

Table 6: Summary of Robertson Stiff ODE System (Example 6)

Item	Details
System Type	Stiff 3-variable ODE (Robertson kinetics)
ODEs	y_1', y_2', y_3' as given
Initial Condition	$y_1(0) = 1,\ y_2(0) = 0,\ y_3(0) = 0$
Rate Constants	$k_1 = 0.04,\ k_2 = 10^4,\ k_3 = 3 \times 10^7$
Time Interval	$t \in [0,\ 4 \times 10^6]$