THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercises on Second-Order Linear Differential Equations

1 Second-Order Linear Differential Equations

A *second-order linear differential equation* is defined as an equation of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \tag{1.1}$$

where P, Q, R, G are continuous functions on a common interval.

This note focuses on the case where $G(x) = 0$ for all x in the domain of Equation (1.1). Such equations are called *homogeneous linear differential equations*, with the standard form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \tag{1.2}$$

If $G(x) \neq 0$ for at least one x, Equation (1.1) is *nonhomogeneous* and is treated in *Additional Topics: Nonhomogeneous Linear Equations.*

Two foundational results enable the solution of homogeneous linear equations. The first is the *superposition principle*, which states that linear combinations of solutions are also solutions:

Theorem 1. *If $y_1(x)$ and $y_2(x)$ are solutions of the linear homogeneous equation (1.2), and c_1, c_2 are arbitrary constants, then the function*

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation (1.2).

Proof. Since y_1 and y_2 satisfy Equation (1.2), we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$

and

$$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0.$$

Using linearity of differentiation, substitute $y = c_1y_1 + c_2y_2$ into the left-hand side of Equation (1.2):

$$\begin{aligned} P(x)y'' + Q(x)y' + R(x)y &= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\ &= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\ &= c_1\left[P(x)y_1'' + Q(x)y_1' + R(x)y_1\right] + c_2\left[P(x)y_2'' + Q(x)y_2' + R(x)y_2\right] \\ &= c_1(0) + c_2(0) = 0. \end{aligned}$$

Thus, $y = c_1y_1 + c_2y_2$ satisfies Equation (1.2). □

The second key result characterizes the general solution using *linearly independent* solutions (proven in advanced courses). Two functions are *linearly independent* on an interval if neither is a constant multiple of the other on that interval. For example, $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, while $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

Theorem 2. *If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of Equation (1.2), and $P(x) \neq 0$ for all x in the domain, then the* general solution *of Equation (1.2) is*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1, c_2 are arbitrary constants.

This theorem is fundamental: if two linearly independent solutions are known, every solution to the equation can be expressed as their linear combination.

For general coefficient functions P, Q, R, finding particular solutions is nontrivial. However, the case of *constant-coefficient homogeneous linear equations* is fully solvable. These equations take the form

$$ay'' + by' + cy = 0 \tag{1.3}$$

where a, b, c are constants with $a \neq 0$.

To solve Equation (1.3), we seek solutions of the form $y = e^{rx}$, where r is a constant. This form is motivated by the fact that the derivative of e^{rx} is a constant multiple of itself: $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Substituting these into Equation (1.3) gives

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Since $e^{rx} \neq 0$ for all real x, this simplifies to the *auxiliary (characteristic) equation*

$$ar^2 + br + c = 0. \tag{1.4}$$

This is an algebraic equation derived from the differential equation by replacing y'' with r^2, y' with r, and y with 1.

The roots of Equation (1.4) can be found via factoring or the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{1.5}$$

The form of the general solution depends on the discriminant $\Delta = b^2 - 4ac$.

Case 1: $\Delta = b^2 - 4ac > 0$ (Real, Distinct Roots)

When the discriminant is positive, the auxiliary equation has two distinct real roots $r_1 \neq r_2$. The functions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are linearly independent (neither is a constant multiple of the other). By Theorem 4, the general solution of Equation (1.3) is:

Theorem 3. *If the roots r_1, r_2 of $ar^2 + br + c = 0$ are real and distinct, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Example 4. Solve $y'' + y' - 6y = 0$.

Solution:

1. Auxiliary Equation: $r^2 + r - 6 = 0$. Factor: $(r-2)(r+3) = 0$, so roots are $r_1 = 2$, $r_2 = -3$.

2. General Solution: By the distinct roots formula,

$$y = c_1 e^{2x} + c_2 e^{-3x}.$$

Example 5. Solve $3\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$.

Solution:

1. Auxiliary Equation: $3r^2 + r - 1 = 0$. Use the quadratic formula:

$$r = \frac{-1 \pm \sqrt{1^2 - 4(3)(-1)}}{2(3)} = \frac{-1 \pm \sqrt{13}}{6}.$$

The roots are real and distinct ($\Delta = 1 + 12 = 13 > 0$).

2. General Solution:

$$y = c_1 e^{\left(\frac{-1+\sqrt{13}}{6}\right)x} + c_2 e^{\left(\frac{-1-\sqrt{13}}{6}\right)x}.$$

Case 2: $\Delta = 0$ (Repeated Real Roots)

When $\Delta = 0$, the auxiliary equation has a single repeated real root $r = -\frac{b}{2a}$. The standard solution $y_1 = e^{rx}$ is one solution; we verify that $y_2 = xe^{rx}$ is a second linearly independent solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx}. \end{aligned}$$

Since $r = -\frac{b}{2a}$, $2ar + b = 0$; since r is a root of the auxiliary equation, $ar^2 + br + c = 0$. Thus $y_2 = xe^{rx}$ satisfies the differential equation. By linear independence, the general solution is:

$$y = c_1 e^{rx} + c_2 xe^{rx}.$$

Theorem 6. *If the auxiliary equation $ar^2 + br + c = 0$ has a repeated real root r, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = c_1 e^{rx} + c_2 xe^{rx}.$$

Example 7. Solve $4y'' + 12y' + 9y = 0$.

Solution:

1. Auxiliary Equation: $4r^2 + 12r + 9 = 0$. Factor: $(2r+3)^2 = 0$, so the repeated root is $r = -\frac{3}{2}$.

2. General Solution: By the repeated roots formula,

$$y = c_1 e^{-3x/2} + c_2 xe^{-3x/2}.$$

Case 3: $\Delta < 0$ (Complex Conjugate Roots)

When $\Delta < 0$, the auxiliary equation has complex conjugate roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where

$$\alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the complex solutions can be rewritten as real-valued solutions:

$$\begin{aligned} y &= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)\right] \\ &= e^{\alpha x}\left[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x\right]. \end{aligned}$$

Letting $c_1 = C_1 + C_2$ and $c_2 = i(C_1 - C_2)$ (real constants), the general solution becomes:

$$y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right).$$

Theorem 8. *If the roots of $ar^2 + br + c = 0$ are complex conjugates $r = \alpha \pm i\beta$, then the general solution of $ay'' + by' + cy = 0$ is*

$$y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right).$$

Example 9. Solve $y'' - 6y' + 13y = 0$.

Solution:

1. Auxiliary Equation: $r^2 - 6r + 13 = 0$. Use the quadratic formula:

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i.$$

Here, $\alpha = 3$, $\beta = 2$.

2. General Solution:

$$y = e^{3x}\left(c_1 \cos 2x + c_2 \sin 2x\right).$$

Initial-Value and Boundary-Value Problems

An *initial-value problem* for a second-order linear differential equation consists of finding a solution $y(x)$ that satisfies the differential equation and two initial conditions at the same point x_0:

$$y(x_0) = y_0, \quad y'(x_0) = y_1,$$

where y_0, y_1 are given constants. If the coefficients $P(x), Q(x), R(x), G(x)$ are continuous on an interval containing x_0, and $P(x) \neq 0$ there, the Existence and Uniqueness Theorem guarantees exactly one solution to this problem.

A *boundary-value problem* requires a solution that satisfies conditions at two different points x_0 and x_1:

$$y(x_0) = y_0, \quad y(x_1) = y_1.$$

Unlike initial-value problems, boundary-value problems do not always have a unique solution (they may have no solution, infinitely many solutions, or one solution).

Initial-Value Problem Examples

Example 10. Solve the initial-value problem:

$$y'' + y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Step-by-Step Solution:

1. General Solution (from Example 1):

The auxiliary equation is $r^2 + r - 6 = 0$, with roots $r_1 = 2, r_2 = -3$. The general solution is:

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}.$$

2. Derivative of the Solution:

Differentiate to apply the initial condition for $y'(0)$:

$$y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}.$$

3. Apply Initial Conditions:

- $y(0) = 1$: Substitute $x = 0$ into $y(x)$:

$$c_1 e^0 + c_2 e^0 = c_1 + c_2 = 1. \tag{1.6}$$

- $y'(0) = 0$: Substitute $x = 0$ into $y'(x)$:

$$2c_1 e^0 - 3c_2 e^0 = 2c_1 - 3c_2 = 0. \tag{1.7}$$

4. Solve the System for c_1, c_2:

From (1.7): $2c_1 = 3c_2 \implies c_2 = \frac{2}{3}c_1$. Substitute into (1.6):

$$c_1 + \frac{2}{3}c_1 = 1 \implies \frac{5}{3}c_1 = 1 \implies c_1 = \frac{3}{5}.$$

Then $c_2 = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$.

5. Final Solution:

$$y(x) = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}.$$

Example 11. Solve the initial-value problem:

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

Step-by-Step Solution:

1. General Solution:

The auxiliary equation is $r^2+1 = 0$, with roots $r = \pm i$ (complex conjugates, $\alpha = 0, \beta = 1$). The general solution is:

$$y(x) = c_1 \cos x + c_2 \sin x.$$

2. Derivative of the Solution:

$$y'(x) = -c_1 \sin x + c_2 \cos x.$$

3. Apply Initial Conditions:

- $y(0) = 2$: Substitute $x = 0$:

$$c_1 \cos 0 + c_2 \sin 0 = c_1 = 2.$$

- $y'(0) = 3$: Substitute $x = 0$:

$$-c_1 \sin 0 + c_2 \cos 0 = c_2 = 3.$$

4. Final Solution:

$$y(x) = 2 \cos x + 3 \sin x.$$

Boundary-Value Problem Example

Example 12. Solve the boundary-value problem:

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y(1) = 3.$$

Step-by-Step Solution:

1. General Solution:

The auxiliary equation is $r^2 + 2r + 1 = 0$, with a repeated root $r = -1$. The general solution is:

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

2. Apply Boundary Conditions:

- $y(0) = 1$: Substitute $x = 0$:

$$c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1 = 1.$$

- $y(1) = 3$: Substitute $x = 1$ and $c_1 = 1$:

$$1 \cdot e^{-1} + c_2 \cdot 1 \cdot e^{-1} = 3.$$

3. Solve for c_2:

Multiply both sides by e to eliminate the exponential:

$$1 + c_2 = 3e \implies c_2 = 3e - 1.$$

4. Final Solution:

$$y(x) = e^{-x} + (3e - 1) x e^{-x}.$$

Discriminant	**Roots of** $ar^2 + br + c = 0$	**General Solution of** $ay'' + by' + cy = 0$
$\Delta = b^2 - 4ac > 0$	Real and distinct roots r_1, r_2	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$\Delta = b^2 - 4ac = 0$	Repeated real root $r = r_1 = r_2$	$y = c_1 e^{rx} + c_2 x e^{rx}$
$\Delta = b^2 - 4ac < 0$	Complex conjugate roots $r = \alpha \pm i\beta$	$y = e^{\alpha x} \left(c_1 \cos \beta x + c_2 \sin \beta x \right)$

Table 1: Complete Solution Summary for Constant-Coefficient Homogeneous Equations

Summary of Solutions to $ay'' + by' + cy = 0$

The table below summarizes the three cases for constant-coefficient homogeneous equations, along with their general solutions:

Exercise

Solve the initial-value problem for $y(t)$:

$$\begin{cases} 2y'' - 3y' + y = 0 \\ y(0) = 2 \\ y'(0) = \dfrac{1}{2} \end{cases}$$

Exercise

Solve the initial-value problem for $y(t)$:

$$\begin{cases} 2y'' - 3y' + y = 0 \\ y(0) = 2 \\ y'(0) = \dfrac{1}{2} \end{cases}$$

1: Find the General Solution

The given differential equation is a second-order linear homogeneous ODE with constant coefficients. We solve it using the auxiliary (characteristic) equation method.

The auxiliary equation for $2y'' - 3y' + y = 0$ is obtained by replacing $y'' \to r^2$, $y' \to r$, and $y \to 1$:

$$2r^2 - 3r + 1 = 0$$

Solve the quadratic equation by factoring:

$$(2r-1)(r-1) = 0$$

The roots are:

$$r_1 = \frac{1}{2}, \quad r_2 = 1$$

Since the roots are real and distinct, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{t/2} + c_2 e^{t}$$

where c_1, c_2 are arbitrary constants.

2: Apply Initial Conditions

We use the initial conditions $y(0) = 2$ and $y'(0) = 1/2$ to solve for c_1 and c_2.

First, compute the derivative of the general solution:

$$y'(t) = \frac{d}{dt}\left(c_1 e^{t/2} + c_2 e^{t}\right) = \frac{1}{2} c_1 e^{t/2} + c_2 e^{t}$$

Now apply the initial conditions:

1. Condition 1: $y(0) = 2$

Substitute $t = 0$ into $y(t)$:

$$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 2 \tag{1}$$

2. Condition 2: $y'(0) = 1/2$

Substitute $t = 0$ into $y'(t)$:

$$y'(0) = \frac{1}{2} c_1 e^0 + c_2 e^0 = \frac{1}{2} c_1 + c_2 = \frac{1}{2} \tag{2}$$

3: Solve the System for c_1, c_2

We solve the linear system (1) and (2):

$$\begin{cases} c_1 + c_2 = 2 \\ \frac{1}{2}c_1 + c_2 = \frac{1}{2} \end{cases}$$

Subtract equation (2) from equation (1) to eliminate c_2:

$$(c_1 + c_2) - \left(\frac{1}{2}c_1 + c_2\right) = 2 - \frac{1}{2}$$

$$\frac{1}{2}c_1 = \frac{3}{2}$$

Multiply both sides by 2:

$$c_1 = 3$$

Substitute $c_1 = 3$ back into equation (1):

$$3 + c_2 = 2 \implies c_2 = -1$$

4: Final Solution

Substitute $c_1 = 3$ and $c_2 = -1$ into the general solution:

$$\boxed{y(t) = 3e^{t/2} - e^t}$$

Verification To confirm the solution is correct, substitute $y(t)$ back into the original differential equation:

- Compute derivatives:

$$y'(t) = \frac{3}{2}e^{t/2} - e^t, \quad y''(t) = \frac{3}{4}e^{t/2} - e^t$$

- Substitute into $2y'' - 3y' + y$:

$$\begin{aligned} & 2\left(\frac{3}{4}e^{t/2} - e^t\right) - 3\left(\frac{3}{2}e^{t/2} - e^t\right) + \left(3e^{t/2} - e^t\right) \\ &= \left(\frac{3}{2}e^{t/2} - 2e^t\right) - \left(\frac{9}{2}e^{t/2} - 3e^t\right) + 3e^{t/2} - e^t \\ &= \left(\frac{3}{2} - \frac{9}{2} + 3\right)e^{t/2} + (-2 + 3 - 1)e^t = 0 \end{aligned}$$

The solution satisfies the differential equation. Checking initial conditions:

- $y(0) = 3e^0 - e^0 = 3 - 1 = 2$

- $y'(0) = \frac{3}{2}e^0 - e^0 = \frac{3}{2} - 1 = \frac{1}{2}$