THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

Exercises on Mathematical Modelling for First-order Ordinary Differential Equations

A first-order ordinary differential equation only involves the first derivative of an unknown function with respect to its independent variable. These equations are widely used in mathematics, physics, engineering and many other scientific fields. This paper examines the mathematical form and solution of two classic applications of linear first-order differential equations:

1 Population Growth and Decay

1.1 Exponential Change

In many real-life situations, how fast a quantity $y(t)$ changes over time t depends on its current value. Examples include radioactive material losing mass and population size changing. These processes are called exponential change.

We want to find the function $y(t)$ that solves this initial value problem (IVP):

$$\frac{dy}{dt} = ky \tag{1}$$

$$y(0) = y_0 \tag{2}$$

where k is a constant.

- If $y > 0$ and the quantity grows, $k > 0$.
- If $y > 0$ and the quantity shrinks, $k < 0$.
- If $y_0 = 0$, then $y(t) = 0$ for all time t (a simple solution to Equation (1)).

When $y \neq 0$, we rearrange Equation (1) to separate variables:

$$\frac{1}{y}\frac{dy}{dt} = k.$$

Integrate both sides with respect to t:

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k\,dt.$$

Using $dy = \frac{dy}{dt}dt$ on the left side:

$$\int \frac{1}{y}\,dy = \int k\,dt.$$

Calculating the integrals gives:

$$\ln|y| = kt + C,$$

where C is an integration constant. We use exponents to solve for y:

$$\begin{aligned} |y| &= e^{kt+C} \\ y &= \pm e^{kt}e^{C}. \end{aligned}$$

Let $A = \pm e^C$. The general solution is:

$$y(t) = Ae^{kt}.$$

If we set $A = 0$, this formula also includes the simple solution $y(t) = 0$.

Now we find A using the initial condition (2). At $t = 0$:

$$y_0 = Ae^0 = A.$$

Substitute $A = y_0$ to get the final solution:

$$y(t) = y_0 e^{kt}. \qquad (3)$$

Equation (3) is exponential growth when $k > 0$ and exponential decay when $k < 0$. The constant k is called the rate constant.

A key feature of radioactive decay is half-life T: the time it takes for a substance to become half its starting amount. For decay ($k < 0$), set $y(T) = \frac{1}{2}y_0$ in Equation (3):

$$\frac{1}{2}y_0 = y_0 e^{-kT}.$$

Divide both sides by y_0 ($y_0 \neq 0$):

$$\frac{1}{2} = e^{-kT}.$$

Take the natural logarithm of both sides:

$$\begin{aligned} \ln\left(\frac{1}{2}\right) &= -kT \\ -\ln 2 &= -kT. \end{aligned}$$

Solve for T:

$$T = \frac{\ln 2}{k}. \qquad (4)$$

1.2 Forward Euler Formula

The forward Euler method (forward scheme) is a simple numerical technique to solve the ordinary differential equation (ODE):

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

Forward Euler Formula

$$y_{n+1} = y_n + dt \cdot k \cdot y_n$$

where

- y_n = current value
- dt = time step
- y_{n+1} = next value

Table 1: Summary of First-Order ODE: Exponential Change

Description	**Equation/Formula**
Original ODE	$\frac{dy}{dt} = ky$
Initial Condition	$y(0) = y_0$
Analytical Solution	$y(t) = y_0 e^{kt}$
Half-Life Formula	$T = \frac{\ln 2}{k}$
Forward Euler Method	$y_{n+1} = y_n + dt \cdot k \cdot y_n$

1.3 Backward Euler Formula

The Backward Euler method (implicit/backward scheme) is an implicit numerical method for solving the ODE:

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

Backward Euler Formula

$$y_{n+1} = y_n + dt \cdot k \cdot y_{n+1}$$

We rearrange it to get an explicit update rule (solved for y_{n+1}):

$$y_{n+1} = \frac{y_n}{1 - k \cdot dt}$$

The backward Euler method is implicit (uses y_{n+1} on both sides), but for the linear ODE $y' = ky$, we can solve it directly:

$$\begin{aligned} y_{n+1} &= y_n + k\,dt\,y_{n+1} \\ y_{n+1} - k\,dt\,y_{n+1} &= y_n \\ y_{n+1}(1 - k\,dt) &= y_n \\ y_{n+1} &= \frac{y_n}{1 - k\,dt} \end{aligned}$$

Table 2: Summary of Backward Euler Method

Item	Formula
Target ODE	$\frac{dy}{dt} = ky,\ y(0) = y_0$
Implicit Formula	$y_{n+1} = y_n + dt \cdot k \cdot y_{n+1}$
Explicit Solved Form	$y_{n+1} = \frac{y_n}{1 - k \cdot dt}$

1.4 Central Difference Scheme

For the ODE

$$\frac{dy}{dt} = ky,$$

the central difference scheme (2nd-order accurate) uses the formula:

$$\frac{y_{n+1} - y_{n-1}}{2dt} = ky_n.$$

Rearranged for the update rule:

$$y_{n+1} = y_{n-1} + 2 \cdot dt \cdot k \cdot y_n.$$

This is a multi-step method (needs two initial values: y_0 and y_1). We use Forward Euler to compute y_1 for initialization.

Table 3: Summary of Central Difference Scheme

Item	Formula
Target ODE	$\frac{dy}{dt} = ky$
Central Difference Formula	$\frac{y_{n+1} - y_{n-1}}{2dt} = ky_n$
Update Rule	$y_{n+1} = y_{n-1} + 2 \cdot dt \cdot k \cdot y_n$
Initialization	Forward Euler for y_1

2 Radioactive Decay

Consider a radioactive substance with a half-life of 1720 years. At time $t = 0$, the mass of the substance is 6 grams. Determine the following:

(a) The remaining mass after 860 years.

(b) The time t_1 at which the remaining mass is 2.5 grams.

Radioactive Decay (Solution)

We define the initial conditions: $t_0 = 0$, initial mass $y_0 = 6$ grams.

The general exponential decay model is:

$$y(t) = 6e^{-kt}. \tag{2.1}$$

The negative exponent indicates radioactive decay ($k > 0$).

Given half-life $T = 1720$ years, the decay constant is:

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{1720}.$$

Substitute k into Equation (2.1):

$$y(t) = 6e^{-\frac{t \ln 2}{1720}}. \tag{2.2}$$

2.1 (a): Mass After 860 Years

Evaluate $y(860)$:

$$\begin{aligned} y(860) &= 6e^{-\frac{860\ln 2}{1720}} \\ &= 6e^{-\frac{\ln 2}{2}} \\ &= 6\left(e^{\ln 2}\right)^{-1/2} \\ &= 6(2)^{-1/2} \\ &= \frac{6}{\sqrt{2}} \\ &= 3\sqrt{2} \\ &\approx 4.24 \text{ grams.} \end{aligned}$$

The remaining mass after 860 years is approximately **4.24** grams.

2.2 (b): Time to Reach 2.5 Grams

Set $y(t_1) = 2.5 = \dfrac{5}{2}$ and substitute into Equation (2.2):

$$\frac{5}{2} = 6e^{-\frac{t_1\ln 2}{1720}}.$$

Divide both sides by 6:

$$\frac{5}{12} = e^{-\frac{t_1\ln 2}{1720}}.$$

Take the natural logarithm of both sides:

$$\ln\left(\frac{5}{12}\right) = -\frac{t_1\ln 2}{1720}.$$

Use the logarithm property $\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$:

$$-\ln\left(\frac{12}{5}\right) = -\frac{t_1\ln 2}{1720}.$$

Cancel negative signs and solve for t_1:

$$\begin{aligned} t_1 &= 1720\cdot\frac{\ln\left(\frac{12}{5}\right)}{\ln 2} \\ &\approx 2172.42 \text{ years.} \end{aligned}$$

The mass reduces to 2.5 grams after approximately **2172.42** years.

3 Population Growth

The population of Zhuhai was 1,108,000 in 2010 and 1,138,000 in 2011. Assume the population grows exponentially at a constant growth rate.

1. Define a variable t to represent time (with $t = 0$ corresponding to 2010) and write the general exponential growth model for the population $y(t)$.
2. Find the constant growth rate k.
3. Use the model to predict the population of Zhuhai in the year 2026.

Population Growth (Solution)

We define $t = 0$ as the year 2010. The initial population is:

$$y_0 = 1108000$$

The general exponential growth model is given by:

$$y(t) = 1108000e^{kt} \tag{3.1}$$

1: Solve for the growth constant k

In 2011 ($t = 1$), the population $y(1) = 1138000$. Substitute into the model:

$$1138000 = 1108000e^{k}$$

Divide both sides by 1108000:

$$e^{k} = \frac{1138000}{1108000} = \frac{1138}{1108}$$

Take the natural logarithm of both sides:

$$k = \ln\left(\frac{1138}{1108}\right) \approx 0.026716$$

2: Final Population Model

Substitute k back into Equation (3.1):

$$y(t) = 1108000e^{0.026716t}$$

3: Predict Population in 2026

For 2026:

$$t = 2026 - 2010 = 15$$

Evaluate $y(15)$:

$$\begin{aligned} y(15) &= 1108000e^{0.026716\cdot 15} \\ &\approx 1108000e^{0.40074} \\ &\approx 1108000 \cdot 1.4924 \\ &\approx 1654165.39 \end{aligned}$$

Rounding to the nearest integer, the predicted population of Zhuhai in 2026 is **1,654,165**.

Table 4: Summary of Radioactive Decay and Population Growth Results

Quantity	**Radioactive Decay**	**Zhuhai Population Growth**
Initial Value	$y_0 = 6$ g	$y_0 = 1\,108\,000$
Half-Life / Growth Rate	$T = 1720$ y, $k = \frac{\ln 2}{1720}$	$k \approx 0.026716$
Mass at 860 years	≈ 4.24 g	—
Time to 2.5 g	≈ 2172.42 y	—
Population (2026, $t = 15$)	—	$\approx 1\,654\,165$