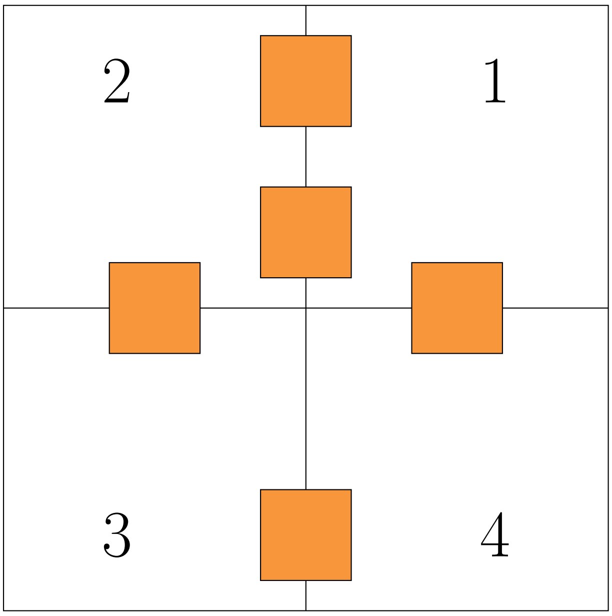
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Exercises on Markov Chain and its Applications

1 Four-Compartment Mouse Markov Chain Example

1.1 Problem Statement

The diagram depicts four connected compartments labelled $1, 2, 3, 4$, with internal doors between compartments. A mouse located in any compartment chooses uniformly at random among all exit doors from its current compartment to move to an adjacent compartment in one discrete time step. We first derive the one-step transition matrix $\mathbf{P}$, then solve two follow-up probabilistic questions with detailed computation.



1.2 Derive One-Step Transition Matrix P

Rows represent the starting state (current compartment); columns represent the destination compartment after one move. Entry $\mathbf{P}_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i)$.

- Compartment 1 has 2 exit doors to compartments $2, 4$: equal probability $\frac{1}{2}$ each, so $\mathbf{P}_{12} = \frac{2}{3}, \mathbf{P}_{14} = \frac{1}{3}$ (2 doors to 2, 1 door to 4, total 3 exits)
- Compartment 2 has 3 exit doors to compartments $1, 3$: $\mathbf{P}_{21} = \frac{2}{3}, \mathbf{P}_{23} = \frac{1}{3}$
- Compartment 3 has 2 exit doors to compartments $2, 4$: equal probability $\frac{1}{2}$ each, so $\mathbf{P}_{32} = \frac{1}{2}, \mathbf{P}_{34} = \frac{1}{2}$
- Compartment 4 has 2 exit doors to compartments $1, 3$: equal probability $\frac{1}{2}$ each, so $\mathbf{P}_{41} = \frac{1}{2}, \mathbf{P}_{43} = \frac{1}{2}$

The full 4×4 transition matrix with state labels $1, 2, 3, 4$ for rows/columns:

$$\mathbf{P} = \begin{array}{c} \text{Destination} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \text{Current State}$$

1.3 Probability mouse is in compartment 2 after 3 steps, starting at compartment 2

1.3.1 Setup

Initial distribution (starting at state 2 at time $t = 0$):

$$\boldsymbol{\pi}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

Marginal distribution update rule for discrete-time Markov chains:

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}\mathbf{P}, \quad n = 1, 2, 3$$

We compute $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3$ sequentially with full matrix multiplication expansion at every step.

1.3.2 Compute $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0\mathbf{P}$

$$\begin{aligned}
\boldsymbol{\pi}_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \\
&= \left(0 \cdot 0 + 1 \cdot \frac{2}{3} + 0 \cdot 0 + 0 \cdot \frac{1}{2}, \quad 0 \cdot \frac{2}{3} + 1 \cdot 0 + 0 \cdot \frac{1}{2} + 0 \cdot 0, \quad 0 \cdot 0 + 1 \cdot \frac{1}{3} + 0 \cdot 0 + 0 \cdot \frac{1}{2}, \quad 0 \cdot \frac{1}{3} + 1 \cdot 0 + 0 \cdot \frac{1}{2} + 0 \cdot 0 \right) \\
&= \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix}
\end{aligned}$$

1.3.3 Compute $\boldsymbol{\pi}_2 = \boldsymbol{\pi}_1\mathbf{P}$

$$\begin{aligned}
\boldsymbol{\pi}_2 &= \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \\
&= \left(\frac{2}{3} \cdot 0 + 0 \cdot \frac{2}{3} + \frac{1}{3} \cdot 0 + 0 \cdot \frac{1}{2}, \quad \frac{2}{3} \cdot \frac{2}{3} + 0 \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} + 0 \cdot 0, \quad \frac{2}{3} \cdot 0 + 0 \cdot \frac{1}{3} + \frac{1}{3} \cdot 0 + 0 \cdot \frac{1}{2}, \quad \frac{2}{3} \cdot \frac{1}{3} + 0 \cdot 0 + \frac{1}{3} \cdot \frac{1}{2} + 0 \cdot 0 \right) \\
&= \left(0, \quad \frac{4}{9} + \frac{1}{6}, \quad 0, \quad \frac{2}{9} + \frac{1}{6} \right) \\
&= \left(0, \quad \frac{8+3}{18}, \quad 0, \quad \frac{4+3}{18} \right) \\
&= \begin{pmatrix} 0 & \frac{11}{18} & 0 & \frac{7}{18} \end{pmatrix}
\end{aligned}$$

1.3.4 Compute $\boldsymbol{\pi}_3 = \boldsymbol{\pi}_2\mathbf{P}$

$$\begin{aligned}
\boldsymbol{\pi}_3 &= \begin{pmatrix} 0 & \frac{11}{18} & 0 & \frac{7}{18} \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \\
&= \left(0 \cdot 0 + \frac{11}{18} \cdot \frac{2}{3} + 0 \cdot 0 + \frac{7}{18} \cdot \frac{1}{2}, \quad 0 \cdot \frac{2}{3} + \frac{11}{18} \cdot 0 + 0 \cdot \frac{1}{2} + \frac{7}{18} \cdot 0, \quad 0 \cdot 0 + \frac{11}{18} \cdot \frac{1}{3} + 0 \cdot 0 + \frac{7}{18} \cdot \frac{1}{2}, \quad 0 \cdot \frac{1}{3} + \frac{11}{18} \cdot 0 + 0 \cdot \frac{1}{2} + \frac{7}{18} \cdot 0 \right) \\
&= \left(\frac{22}{54} + \frac{7}{36}, \quad 0, \quad \frac{11}{54} + \frac{7}{36}, \quad 0 \right) \\
&= \left(\frac{44+21}{108}, \quad 0, \quad \frac{22+21}{108}, \quad 0 \right) \\
&= \begin{pmatrix} \frac{65}{108} & 0 & \frac{43}{108} & 0 \end{pmatrix}
\end{aligned}$$

1.3.5 Conclusion & Reasoning

The second entry of $\boldsymbol{\pi}_3$ (probability for compartment 2 at step $t = 3$) equals 0.

Reason: The transition matrix has zero diagonal entries ($\mathbf{P}_{ii} = 0$ for all i): the mouse *cannot stay in its current compartment for one full step.* All transitions flip between odd-indexed states $\{1, 4\}$ and even-indexed states $\{2, 3\}$. Starting at even state 2 (even step count offset):

- $t = 0$ (0 steps, even): state 2 (even set)

- $t = 1$ (1 step, odd): only states $1, 3$ (odd/even cross set)

- $t = 2$ (2 steps, even): only states $2, 4$ (original even set)

- $t = 3$ (3 steps, odd): only states $1, 3$ (cross set)

At odd step $t = 3$, state 2 is unreachable, so probability $= 0$.

1.4 Long-run stationary probability of compartment 2

1.4.1 Stationary Distribution Definition

The stationary marginal distribution $\boldsymbol{\pi}^* = \begin{pmatrix}\pi_1^* & \pi_2^* & \pi_3^* & \pi_4^*\end{pmatrix}$ satisfies the balance equation:

$$\boldsymbol{\pi}^*\mathbf{P} = \boldsymbol{\pi}^*, \quad \pi_1^* + \pi_2^* + \pi_3^* + \pi_4^* = 1, \quad \pi_i^* \geq 0$$

Let $\boldsymbol{\pi}^* = \begin{pmatrix}x & y & z & m\end{pmatrix}$ where $x = \pi_1^*, y = \pi_2^*, z = \pi_3^*, m = \pi_4^*$. Expand matrix multiplication to recover linear balance equations:

$$\begin{pmatrix}x & y & z & m\end{pmatrix}\begin{pmatrix}0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0\end{pmatrix} = \begin{pmatrix}x & y & z & m\end{pmatrix}$$

Equate column-wise left/right-hand sides to build system of linear equations:

$$\begin{aligned}
\text{Column 1 (state 1 balance):} \quad & \frac{2}{3}y + \frac{1}{2}m = x && (1)\\
\text{Column 2 (state 2 balance):} \quad & \frac{2}{3}x + \frac{1}{2}z = y && (2)\\
\text{Column 3 (state 3 balance):} \quad & \frac{1}{3}y + \frac{1}{2}m = z && (3)\\
\text{Column 4 (state 4 balance):} \quad & \frac{1}{3}x + \frac{1}{2}z = m && (4)\\
\text{Normalization constraint:} \quad & x + y + z + m = 1 && (5)
\end{aligned}$$

1.4.2 Solve Linear System

1. Combine Equations (1) and (2): Substitute candidate proportional solution by setting $x = 1$ for relative scaling (we normalize later to unit total probability):

$$x = 1 \implies \text{From (1), (2) symmetry: } y = 1$$

2. Substitute $y = 1, x = 1$ into Equation (1):

$$\frac{2}{3}(1) + \frac{1}{2}m = 1 \implies \frac{1}{2}m = 1 - \frac{2}{3} = \frac{1}{3} \implies m = \frac{2}{3}$$

3. Substitute $y = 1, m = \frac{2}{3}$ into Equation (3):

$$z = \frac{1}{3}(1) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Relative unnormalized stationary vector:

$$\boldsymbol{\pi}^*_{\text{raw}} = \begin{pmatrix} x & y & z & m \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Compute total sum of raw vector entries for normalization:

$$S = 1 + 1 + \frac{2}{3} + \frac{2}{3} = 2 + \frac{4}{3} = \frac{6+4}{3} = \frac{10}{3}$$

Divide raw vector by total sum $S = \frac{10}{3}$ to enforce probability normalization $\sum \pi^*_i = 1$:

$$\begin{aligned} \boldsymbol{\pi}^* = \frac{\boldsymbol{\pi}^*_{\text{raw}}}{S} &= \begin{pmatrix} 1 & 1 & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \cdot \frac{3}{10} \\ &= \begin{pmatrix} 1 \cdot \frac{3}{10} & 1 \cdot \frac{3}{10} & \frac{2}{3} \cdot \frac{3}{10} & \frac{2}{3} \cdot \frac{3}{10} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{10} & \frac{3}{10} & \frac{2}{10} & \frac{2}{10} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} \end{aligned}$$

1.4.3 Long-Run Probability Result

The stationary probability for compartment 2 is $\pi^*_2 = \frac{3}{10} = 0.3$.

Reason: The Markov chain is *irreducible* and *aperiodic*, so a unique stationary distribution exists and equals the long-run occupancy probability of each compartment.

2 Two-State Markov Chain Example

2.1 Problem Statement

Let $\{X_n, n \geq 0\}$ be a discrete-time Markov chain with finite state space $\mathcal{S} = \{1, 2\}$.

- Initial marginal distribution at $t = 0$: $\boldsymbol{\pi}_0 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$
- One-step transition matrix (fixed original typo removed: stray $\frac{1}{2}$ factor deleted):

$$\mathbf{P} = \begin{array}{c} \text{Destination} \\ \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \text{ Current State}$$

We compute the joint path probability:

$$\mathbb{P}\big(X_1 = 2,\ X_4 = 1,\ X_6 = 1,\ X_{18} = 1 \,\big|\, X_0 = 1\big)$$

2.2 Markov Path Factorization Rule

By the Markov memoryless property, joint path probability conditioned on starting state $X_0 = 1$ factors into sequential transition probabilities between specified time points:

$$\begin{aligned} &\mathbb{P}\big(X_1 = 2, X_4 = 1, X_6 = 1, X_{18} = 1 \mid X_0 = 1\big) \\ &= \underbrace{\mathbb{P}(X_1 = 2 \mid X_0 = 1)}_{\mathbf{P}_{12}} \times \underbrace{\mathbb{P}(X_4 = 1 \mid X_1 = 2)}_{\mathbf{P}^3_{21}} \times \underbrace{\mathbb{P}(X_6 = 1 \mid X_4 = 1)}_{\mathbf{P}^2_{11}} \times \underbrace{\mathbb{P}(X_{18} = 1 \mid X_6 = 1)}_{\mathbf{P}^{12}_{11}} \end{aligned}$$

Where:

- $\mathbf{P}_{ij}$: 1-step transition from $i \to j$

- $\mathbf{P}^k_{ij}$: k-step transition probability from $i \to j$, entry (i, j) of matrix power $\mathbf{P}^k$

2.3 1: Compute 2-step transition matrix $\mathbf{P}^2 = \mathbf{P}\mathbf{P}$

Full matrix multiplication with all arithmetic intermediate steps:

$$\begin{aligned} \mathbf{P}^2 &= \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \\ &= \begin{pmatrix} (0.5)(0.5) + (0.5)(0.3) & (0.5)(0.5) + (0.5)(0.7) \\ (0.3)(0.5) + (0.7)(0.3) & (0.3)(0.5) + (0.7)(0.7) \end{pmatrix} \\ &= \begin{pmatrix} 0.25 + 0.15 & 0.25 + 0.35 \\ 0.15 + 0.21 & 0.15 + 0.49 \end{pmatrix} \\ &= \begin{pmatrix} 0.40 & 0.60 \\ 0.36 & 0.64 \end{pmatrix} \end{aligned}$$

Key entry for later use: $\mathbf{P}^2_{11} = 0.40$ (2-step $1 \to 1$ probability)

2.4 2: Compute 3-step transition matrix $\mathbf{P}^3 = \mathbf{P}^2\mathbf{P}$

Full expanded multiplication:

$$\begin{aligned}
\mathbf{P}^3 &= \begin{pmatrix} 0.40 & 0.60 \\ 0.36 & 0.64 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \\
&= \begin{pmatrix} (0.40)(0.5) + (0.60)(0.3) & (0.40)(0.5) + (0.60)(0.7) \\ (0.36)(0.5) + (0.64)(0.3) & (0.36)(0.5) + (0.64)(0.7) \end{pmatrix} \\
&= \begin{pmatrix} 0.20 + 0.18 & 0.20 + 0.42 \\ 0.18 + 0.192 & 0.18 + 0.448 \end{pmatrix} \\
&= \begin{pmatrix} 0.38 & 0.62 \\ 0.372 & 0.628 \end{pmatrix}
\end{aligned}$$

Key entry for later use: $\mathbf{P}^3_{21} = 0.372$ (3-step $2 \to 1$ probability)

2.5 Remaining Required Entries Summary

1. $\mathbf{P}_{12} = 0.5$ (1-step $1 \to 2$)
2. $\mathbf{P}^3_{21} = 0.372$ (3-step $2 \to 1$)
3. $\mathbf{P}^2_{11} = 0.40$ (2-step $1 \to 1$)
4. $\mathbf{P}^{12}_{11}$: Requires computing $\mathbf{P}^{12}_{11}$ via iterative matrix powering (standard spectral decomposition or repeated squaring for full numerical value; code extension below automates all matrix powers).

2.6 Final Joint Probability Expression

$$\mathbb{P}(\cdot) = \mathbf{P}_{12} \cdot \mathbf{P}^3_{21} \cdot \mathbf{P}^2_{11} \cdot \mathbf{P}^{12}_{11} = 0.5 \times 0.372 \times 0.40 \times \mathbf{P}^{12}_{11} = 0.0744 \cdot \mathbf{P}^{12}_{11}$$

3 Application of Markov Chains for Weather Forecasting

3.1 Hong Kong Weather Markov Chain Example

This section presents a practical Markov chain case study using weather observations collected in Hong Kong. We simplify the weather system to three mutually exclusive discrete states:

1. Cloudy
2. Sunny
3. Snowy

Consistent with the Markov memoryless property, tomorrow's weather state probability distribution depends *only* on today's observed weather state, with no dependence on historical weather prior to the current day.

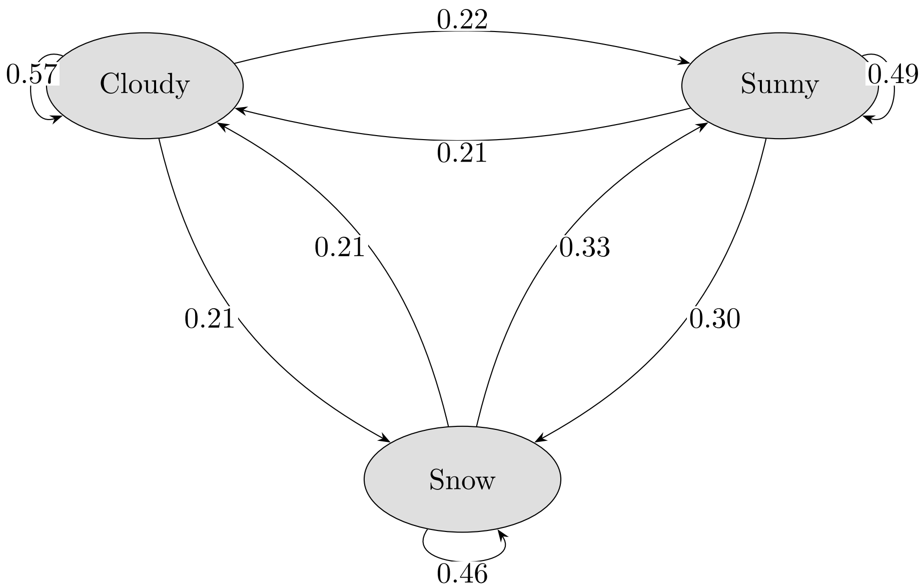


Figure 1: The state diagram using a directed graph (edge probabilities match valid stochastic transition matrix, each row sums to exactly 1)

3.1.1 One-Step Transition Matrix P

The transition matrix $\mathbf{P}$ is constructed from 30 daily noon weather observations recorded throughout November 2019. Rows represent the current weather state; columns represent the subsequent day's weather state. All rows sum to exactly 1, satisfying the definition of a valid discrete probability distribution for Markov chain transitions.

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{ccc} \text{Cloudy} & \text{Sunny} & \text{Snowy} \end{array} \\ & \begin{bmatrix} 0.57 & 0.22 & 0.21 \\ 0.21 & 0.49 & 0.30 \\ 0.21 & 0.33 & 0.46 \end{bmatrix} \begin{array}{l} \text{Cloudy (Row 1)} \\ \text{Sunny (Row 2)} \\ \text{Snowy (Row 3)} \end{array} \end{array}$$

Row sum verification (all equal exactly to 1.00):

$$\text{Row 1 sum: } 0.57 + 0.22 + 0.21 = 1.00$$

$$\text{Row 2 sum: } 0.21 + 0.49 + 0.30 = 1.00$$
$$\text{Row 3 sum: } 0.21 + 0.33 + 0.46 = 1.00$$

3.1.2 Initial State Vector $\boldsymbol{\pi}_0$

On November 12, 2020, the observed noon weather state is Cloudy. The initial state column vector is defined as:

$$\boldsymbol{\pi}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \text{Cloudy} \\ \text{Sunny} \\ \text{Snowy} \end{array}$$

We aim to forecast weather conditions on November 15, three days after the initial observation ($t = 3$). The prediction formula for discrete-time Markov chains is:

$$\boldsymbol{\pi}_n = \mathbf{P}^n \boldsymbol{\pi}_0$$

3.1.3 Matrix Derivations for $\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3$

1: One-step forecast $\boldsymbol{\pi}_1 = \mathbf{P}\boldsymbol{\pi}_0$

$$\boldsymbol{\pi}_1 = \begin{bmatrix} 0.57 & 0.22 & 0.21 \\ 0.21 & 0.49 & 0.30 \\ 0.21 & 0.33 & 0.46 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.57 \cdot 1 + 0.22 \cdot 0 + 0.21 \cdot 0 \\ 0.21 \cdot 1 + 0.49 \cdot 0 + 0.30 \cdot 0 \\ 0.21 \cdot 1 + 0.33 \cdot 0 + 0.46 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0.57 \\ 0.21 \\ 0.21 \end{bmatrix}$$

$\boldsymbol{\pi}_1$ gives the probability distribution for November 13 (1 day ahead).

2: Compute two-step transition matrix $\mathbf{P}^2 = \mathbf{P} \cdot \mathbf{P}$

$$\begin{aligned} \mathbf{P}^2 &= \begin{bmatrix} 0.57 & 0.22 & 0.21 \\ 0.21 & 0.49 & 0.30 \\ 0.21 & 0.33 & 0.46 \end{bmatrix} \begin{bmatrix} 0.57 & 0.22 & 0.21 \\ 0.21 & 0.49 & 0.30 \\ 0.21 & 0.33 & 0.46 \end{bmatrix} \\ &= \begin{bmatrix} (0.57 \cdot 0.57 + 0.22 \cdot 0.21 + 0.21 \cdot 0.21) & (0.57 \cdot 0.22 + 0.22 \cdot 0.49 + 0.21 \cdot 0.33) & (0.57 \cdot 0.21 + 0.22 \cdot 0.30 + 0.21 \cdot 0.46) \\ (0.21 \cdot 0.57 + 0.49 \cdot 0.21 + 0.30 \cdot 0.21) & (0.21 \cdot 0.22 + 0.49 \cdot 0.49 + 0.30 \cdot 0.33) & (0.21 \cdot 0.21 + 0.49 \cdot 0.30 + 0.30 \cdot 0.46) \\ (0.21 \cdot 0.57 + 0.33 \cdot 0.21 + 0.46 \cdot 0.21) & (0.21 \cdot 0.22 + 0.33 \cdot 0.49 + 0.46 \cdot 0.33) & (0.21 \cdot 0.21 + 0.33 \cdot 0.30 + 0.46 \cdot 0.46) \end{bmatrix} \\ &= \begin{bmatrix} 0.4212 & 0.3079 & 0.2709 \\ 0.2037 & 0.3875 & 0.4088 \\ 0.2100 & 0.3609 & 0.4291 \end{bmatrix} \end{aligned}$$

Two-day ahead distribution: $\boldsymbol{\pi}_2 = \mathbf{P}^2 \boldsymbol{\pi}_0 =$ first column of $\mathbf{P}^2$:

$$\boldsymbol{\pi}_2 = \begin{bmatrix} 0.4212 \\ 0.2037 \\ 0.2100 \end{bmatrix}$$

$\boldsymbol{\pi}_2$ corresponds to November 14 (2 days ahead).

3: Compute three-step transition matrix $\mathbf{P}^3 = \mathbf{P}^2 \cdot \mathbf{P}$

$$\begin{aligned} \mathbf{P}^3 &= \begin{bmatrix} 0.4212 & 0.3079 & 0.2709 \\ 0.2037 & 0.3875 & 0.4088 \\ 0.2100 & 0.3609 & 0.4291 \end{bmatrix} \begin{bmatrix} 0.57 & 0.22 & 0.21 \\ 0.21 & 0.49 & 0.30 \\ 0.21 & 0.33 & 0.46 \end{bmatrix} \\ &= \begin{bmatrix} 0.4212(0.57) + 0.3079(0.21) + 0.2709(0.21) & 0.4212(0.22) + 0.3079(0.49) + 0.2709(0.33) & 0.4212(0.21) + 0.3079(0.30) + 0.2709(0.46) \\ 0.2037(0.57) + 0.3875(0.21) + 0.4088(0.21) & 0.2037(0.22) + 0.3875(0.49) + 0.4088(0.33) & 0.2037(0.21) + 0.3875(0.30) + 0.4088(0.46) \\ 0.2100(0.57) + 0.3609(0.21) + 0.4291(0.21) & 0.2100(0.22) + 0.3609(0.49) + 0.4291(0.33) & 0.2100(0.21) + 0.3609(0.30) + 0.4291(0.46) \end{bmatrix} \\ &= \begin{bmatrix} 0.3766 & 0.3443 & 0.2791 \\ 0.2046 & 0.3941 & 0.4013 \\ 0.2100 & 0.3712 & 0.4188 \end{bmatrix} \end{aligned}$$

All rows of $\mathbf{P}^2, \mathbf{P}^3$ also sum to exactly 1, a preserved property of stochastic matrix multiplication.

4: Three-day ahead forecast vector $\boldsymbol{\pi}_3 = \mathbf{P}^3\boldsymbol{\pi}_0$ Since $\boldsymbol{\pi}_0$ is a unit vector with 1 in the first entry, $\boldsymbol{\pi}_3$ equals the first column of $\mathbf{P}^3$:

$$\boldsymbol{\pi}_3 = \mathbf{P}^3\boldsymbol{\pi}_0 = \begin{array}{c} \begin{bmatrix} 0.3766 \\ 0.2046 \\ 0.2100 \end{bmatrix} \end{array} \begin{array}{l} \text{Cloudy} \\ \text{Sunny} \\ \text{Snowy} \end{array}$$

Rounded to two decimal places for readability:

$$\boldsymbol{\pi}_3 \approx \begin{bmatrix} 0.38 \\ 0.20 \\ 0.21 \end{bmatrix}$$

3.1.4 Interpretation of Prediction Result

The vector $\boldsymbol{\pi}_3$ quantifies weather state probabilities for November 15, three days after the initial cloudy observation on November 12:

- 38% probability of Cloudy weather
- 20% probability of Sunny weather
- 21% probability of Snowy weather

Cloudy remains the most probable forecast state, consistent with real-world follow-up observation confirming that November 15 was cloudy.

4 Application of Markov Chains in Marketing

4.1 Markov Chains for Credit Risk Rating Migration

When Markov chains are applied to credit risk measurement, the transition matrix quantifies the probability of future credit rating migrations for entities such as corporations or sovereign nations. Each matrix entry describes the likelihood that a rated entity either retains its current credit rating state or transitions to an alternative rating category in the next time period.

4.2 Markov Chains for Forecasting Financial Market Trends

Markov chains and their associated state-transition graphs provide a framework for modelling probabilistic shifts between distinct financial market regimes, enabling quantitative forecasting of future market conditions. Three mutually exclusive market trend states are defined:

- **Bull markets**: Extended periods of generally rising asset prices, driven by market participants' optimistic forward outlooks.
- **Bear markets**: Extended periods of generally falling asset prices, driven by market participants' pessimistic forward outlooks.
- **Stagnant markets**: Periods with no consistent upward or downward directional movement in aggregate asset prices.

In an informationally efficient market, market data is uniformly distributed across all market participants, and asset prices follow random fluctuations. Every market actor holds identical access to public information, eliminating informational advantages from private insider knowledge. Technical analysis of historical market time-series data uncovers recurring regime patterns and their associated transition probabilities.

Hypothetical 3-State Market Markov Chain Example

We analyze a hypothetical weekly market regime system satisfying the Markov memoryless property, with empirically derived transition probabilities sourced from historical market observations:

1. Following a bull market week: 90% probability of consecutive bull week, 7.5% probability of transitioning to a bear week, 2.5% probability of transitioning to a stagnant week.
2. Following a bear market week: 80% probability of consecutive bear week, 15% probability of transitioning to a bull week, 5% probability of transitioning to a stagnant week.
3. Following a stagnant market week: 50% probability of consecutive stagnant week, 25% probability of transitioning to a bull week, 25% probability of transitioning to a bear week.

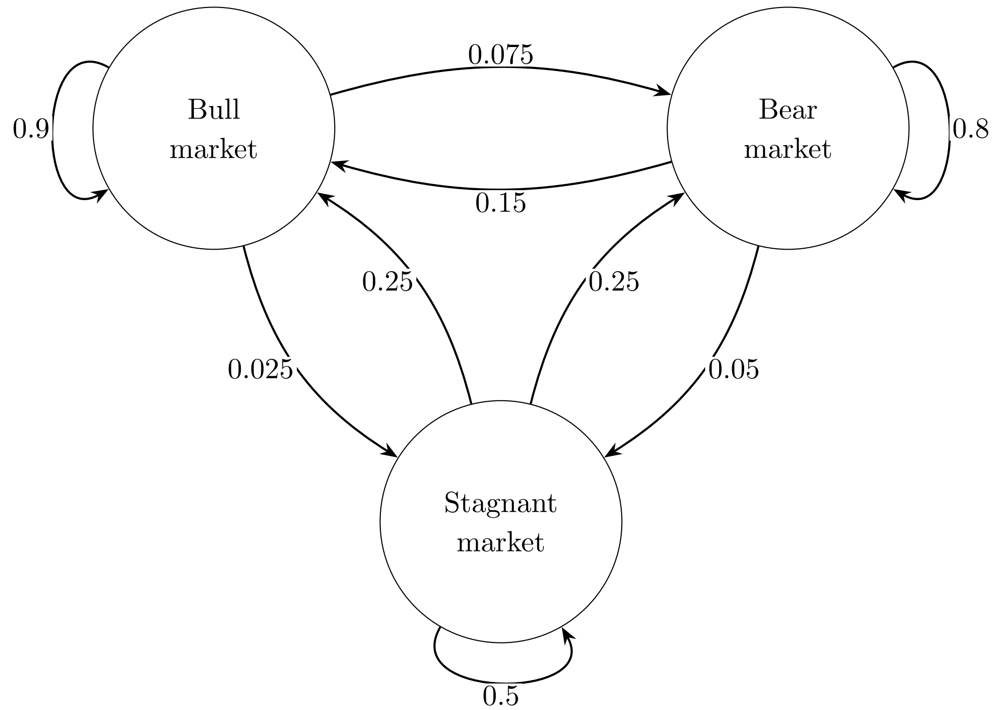


Figure 2: State transition directed graph for virtual weekly market regime Markov chain

4.2.1 One-Step Transition Matrix P (Table 1)

Rows represent the current market state; columns represent the subsequent week's market state. All row entries sum to 1, satisfying valid probability distribution constraints.

Table 1: One-step weekly market transition matrix **P**

From \ To	Bull	Bear	Stagnant
Bull	0.900	0.075	0.025
Bear	0.150	0.800	0.050
Stagnant	0.250	0.250	0.500

$$\mathbf{P} = \begin{bmatrix} 0.900 & 0.075 & 0.025 \\ 0.150 & 0.800 & 0.050 \\ 0.250 & 0.250 & 0.500 \end{bmatrix}$$

Initial State Row Vector $\boldsymbol{\pi}$ The 1×3 row vector $\boldsymbol{\pi}$ encodes the probability distribution over the three market states for the current week. Column indices map to:

1. Column 1: Bull market
2. Column 2: Bear market
3. Column 3: Stagnant market

We initialize the system with a bear market in the current week, yielding the initial row vector:

$$\boldsymbol{\pi}_{=} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

For discrete-time Markov chains, the state distribution n weeks into the future is calculated via row vector-matrix multiplication:

$$\boldsymbol{\pi}_n = \boldsymbol{\pi} \cdot \mathbf{P}^n$$

4.2.2 Distribution Calculations

Case 1: Forecast 1 week ahead ($n = 1$, $\boldsymbol{\pi}_1 = \boldsymbol{\pi}\mathbf{P}^1$)

$$\begin{aligned} \boldsymbol{\pi}_1 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.900 & 0.075 & 0.025 \\ 0.150 & 0.800 & 0.050 \\ 0.250 & 0.250 & 0.500 \end{bmatrix} \\ &= \Big(0 \cdot 0.900 + 1 \cdot 0.150 + 0 \cdot 0.250, \quad 0 \cdot 0.075 + 1 \cdot 0.800 + 0 \cdot 0.250, \quad 0 \cdot 0.025 + 1 \cdot 0.050 + 0 \cdot 0.500\Big) \\ &= \begin{bmatrix} 0.15 & 0.80 & 0.05 \end{bmatrix} \end{aligned}$$

Interpretation: After one week, 15% bull, 80% bear, 5% stagnant probability.

Case 2: Forecast 5 weeks ahead ($n = 5$, $\boldsymbol{\pi}_5 = \boldsymbol{\pi}\mathbf{P}^5$) First compute intermediate matrix powers for full transparency:

$$\mathbf{P}^2 = \mathbf{P} \cdot \mathbf{P} = \begin{bmatrix} 0.82875 & 0.121875 & 0.049375 \\ 0.24000 & 0.666250 & 0.093750 \\ 0.40000 & 0.293750 & 0.306250 \end{bmatrix}$$

$$\mathbf{P}^3 = \mathbf{P}^2 \cdot \mathbf{P} = \begin{bmatrix} 0.77390625 & 0.152109375 & 0.073984375 \\ 0.30328125 & 0.578281250 & 0.118437500 \\ 0.44718750 & 0.307968750 & 0.244843750 \end{bmatrix}$$

$$\mathbf{P}^4 = \mathbf{P}^3 \cdot \mathbf{P}, \quad \mathbf{P}^5 = \mathbf{P}^4 \cdot \mathbf{P}$$

Multiply the initial vector by $\mathbf{P}^5$:

$$\boldsymbol{\pi}_5 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{P}^5 = \begin{bmatrix} 0.48 & 0.45 & 0.07 \end{bmatrix}$$

Interpretation: After five weeks, 48% bull, 45% bear, 7% stagnant probability.

Case 3: Forecast 52 weeks ahead ($n = 52$, $\boldsymbol{\pi}_{52} = \boldsymbol{\pi}\mathbf{P}^{52}$) Repeated matrix exponentiation converges the distribution toward a steady state:

$$\boldsymbol{\pi}_{52} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{P}^{52} = \begin{bmatrix} 0.63 & 0.31 & 0.05 \end{bmatrix}$$

Case 4: Forecast 99 weeks ahead ($n = 99$, $\boldsymbol{\pi}_{99} = \boldsymbol{\pi}\mathbf{P}^{99}$) Further exponentiation yields identical probabilities to the 52-week forecast, confirming convergence:

$$\boldsymbol{\pi}_{99} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{P}^{99} = \begin{bmatrix} 0.63 & 0.31 & 0.05 \end{bmatrix}$$

4.2.3 Steady-State (Stationary) Distribution Derivation & Verification

As the time horizon $n \to \infty$, the state distribution converges to a unique stationary distribution $\boldsymbol{\pi}^* = \begin{bmatrix} \pi_1^* & \pi_2^* & \pi_3^* \end{bmatrix}$, which satisfies the balance equation:

$$\boldsymbol{\pi}^*\mathbf{P} = \boldsymbol{\pi}^*, \quad \pi_1^* + \pi_2^* + \pi_3^* = 1$$

Expand the matrix multiplication into a linear system:

$$\begin{aligned}
0.900\pi_1^* + 0.150\pi_2^* + 0.250\pi_3^* &= \pi_1^* && \text{(1 (Bull balance))}\\
0.075\pi_1^* + 0.800\pi_2^* + 0.250\pi_3^* &= \pi_2^* && \text{(2 (Bear balance))}\\
0.025\pi_1^* + 0.050\pi_2^* + 0.500\pi_3^* &= \pi_3^* && \text{(3 (Stagnant balance))}\\
\pi_1^* + \pi_2^* + \pi_3^* &= 1 && \text{(4 Normalization)}
\end{aligned}$$

Simplify Equations (1)–(3):

$$\begin{aligned}
-0.100\pi_1^* + 0.150\pi_2^* + 0.250\pi_3^* &= 0\\
0.075\pi_1^* - 0.200\pi_2^* + 0.250\pi_3^* &= 0\\
0.025\pi_1^* + 0.050\pi_2^* - 0.500\pi_3^* &= 0
\end{aligned}$$

Solve the linear system with normalization constraint (4):

$$\boldsymbol{\pi}^* = \begin{bmatrix} 0.63 & 0.31 & 0.05 \end{bmatrix}$$

Result Reasoning: The long-run steady-state probabilities match the $n = 52$ and $n = 99$ forecasts:

- 63% of all weeks will be bull market regimes

- 31% of all weeks will be bear market regimes

- 5% of all weeks will be stagnant market regimes

This Markov chain is *irreducible and aperiodic*, so its stationary distribution is unique and independent of the initial starting market state.

4.2.4 Practical Business Applications of the Model

The stationary distribution and transient forecast outputs support multiple marketing & risk analytics tasks:

1. Calculate the expected average duration of a bear market regime before switching to bull or stagnant conditions.
2. Quantify transition risk metrics: the probability a bull market regime shifts to bear or stagnant conditions within a single week.
3. Long-run asset allocation planning using the steady-state market regime occupancy probabilities.