EDB Math Modelling Workshop (Part II)

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content

What is a System of ODEs?

- Mathematical description of interacting quantities:

$$\begin{cases} \frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n) \\ \quad \vdots \\ \frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

Matrix-Vector Representation

Compact Notation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

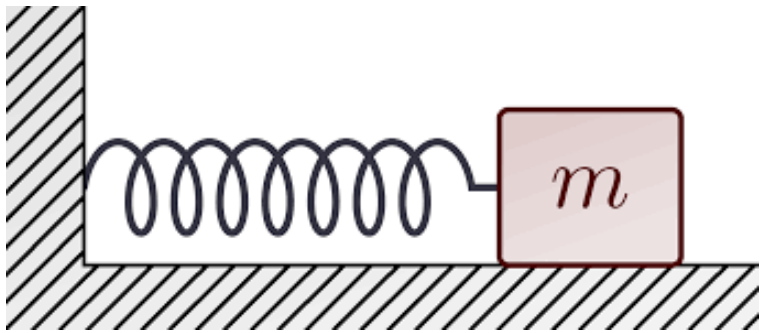
Modeling Paradigm & Example

From Reality to Equations:

Case Study: Spring-Mass-Damper System

Physical System

- Mass m attached to spring (k) and damper (c)
- Displacement $x(t)$ from equilibrium



Spring-Mass-Damper Model Equations

Modeling Steps

1. Newton's 2nd Law:
$$m\ddot{x} = -c\dot{x} - kx$$
2. Introduce state variables: $x_1 = x$, $x_2 = \dot{x}$
3. Convert to system:
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{cases}$$

Matrix Form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Fundamental Classifications of ODE Systems

System Types

- **Linear/Nonlinear**:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \quad vs \quad \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

- **Autonomous/Non-autonomous**:

$$\mathbf{f}(\mathbf{x}) \quad vs \quad \mathbf{f}(t, \mathbf{x})$$

- **Constant/Variable coefficients**

Reality

Most real-world systems are:

- Nonlinear
- High-dimensional
- Require numerical solutions

Domain Applications

Biology/Ecology

- Predator-Prey Dynamics:

$$\begin{cases} \frac{dR}{dt} = \alpha R - \beta RF \\ \frac{dF}{dt} = \delta RF - \gamma F \end{cases}$$

Epidemiology

- SIR Model:

$$\begin{cases} \frac{dS}{dt} = -\beta SI \\ \frac{dI}{dt} = \beta SI - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases}$$

Universal Framework

Applicability

The framework of ODE systems applies broadly:

- Engineering systems
- Economic models
- Climate and environmental models
- Neural and biological systems

From Scalar ODE to Systems

- Consider a first-order scalar ODE:

$$x'(t) = ax(t)$$

- Try a solution of the form: $x(t) = e^{\lambda t}u$

$$\Rightarrow x'(t) = \lambda e^{\lambda t}u = ae^{\lambda t}u \Rightarrow \lambda = a$$

- Now consider a system:

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

- Can we try a similar form? Yes!

$$\mathbf{x}(t) = \mathbf{u}e^{\lambda t} \Rightarrow A\mathbf{u} = \lambda\mathbf{u}$$

- Leads to the eigenvalue problem for matrix A

Details of the Derivation

From Assumption to Eigenvalue Equation

Step 1: Assume Exponential Solution

$$\mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = e^{\lambda t}\mathbf{u}$$

Step 2: Differentiate Solution

$$\frac{d\mathbf{x}}{dt} = \lambda e^{\lambda t}\mathbf{u}$$

Step 3: Substitute into System

$$\lambda e^{\lambda t}\mathbf{u} = A(e^{\lambda t}\mathbf{u})$$

Step 4: Simplify

$$\lambda\mathbf{u} = A\mathbf{u} \Rightarrow (A - \lambda I)\mathbf{u} = \mathbf{0}$$

Critical Equation

Characteristic Equation

Non-trivial solutions require:

$$\det(A - \lambda I) = 0$$

2x2 Case Demonstration

For matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Solutions:

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

- Roots determine solution behavior
- Complex/Repeated roots require special handling

Example(as exercise)

The initial condition would be given later.

Given System:

$$\begin{cases} \frac{dx_1}{dt} = 2x_1 - 2x_2 \\ \frac{dx_2}{dt} = 2x_1 - 3x_2 \end{cases}$$

Matrix Representation:

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}$$

Characteristic Equation:

$$\det \begin{bmatrix} 2-\lambda & -2 \\ 2 & -3-\lambda \end{bmatrix} = \lambda^2 + \lambda - 2 = 0$$

Eigenvalue Problem

Roots Calculation:

$$\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

Eigenvector Determination

For $\lambda_1 = 1$:

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{0}$$

Solution: $u_1 = 2u_2$

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$:

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}$$

Solution: $v_1 = \frac{1}{2} v_2$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Superposition Principle

Theorem (Superposition Principle)

If $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_k(t)$ *are solutions to the homogeneous linear system*

$$\mathbf{x}' = A\mathbf{x},$$

then any linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution, where $c_1, \ldots, c_k \in \mathbb{R}$.

Remark

This principle allows us to build the general solution by combining basic solutions.

- *Helps us construct solutions from eigen-solutions.*

Eigenvalue Method for Linear Systems

Theorem (General Solution of a Linear System)

Suppose $A \in \mathbb{R}^{n \times n}$ has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the general solution of the system

$$\mathbf{x}' = A\mathbf{x}$$

is given by

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t},$$

where $c_1, \ldots, c_n \in \mathbb{R}$ are constants determined by initial conditions.

Remark

Each term $\mathbf{v}_i e^{\lambda_i t}$ represents a fundamental mode of the system. The sign and nature (real/complex) of λ_i determines the behavior:

- $\lambda_i > 0$: *exponential growth*
- $\lambda_i < 0$: *exponential decay*
- $\lambda_i \in \mathbb{C}$: *oscillation (spiral behavior)*

Solution Construction

Theorem (General Solution Formula)

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2$$

Applied to Our Case:

$$\mathbf{x}(t) = c_1 e^{t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}1\\2\end{bmatrix}$$

Component Form

$$\begin{cases} x_1(t) = 2c_1 e^{t} + c_2 e^{-2t} \\ x_2(t) = c_1 e^{t} + 2c_2 e^{-2t} \end{cases}$$

Determining Constants with the Initial Condition

Given Initial Condition:

$$\mathbf{x}(0) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

System of Equations:

$$\begin{cases} 2c_1 + c_2 = -1 \\ c_1 + 2c_2 = 4 \end{cases}$$

Matrix Form:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Solution:

$$c_1 = -2, \quad c_2 = 3$$

Final Solution Expression

Complete Particular Solution

$$\mathbf{x}(t) = -2e^{t}\begin{bmatrix}2\\1\end{bmatrix} + 3e^{-2t}\begin{bmatrix}1\\2\end{bmatrix}$$

Component Form:

$$\begin{cases} x_1(t) = -4e^{t} + 3e^{-2t} \\ x_2(t) = -2e^{t} + 6e^{-2t} \end{cases}$$

- Demonstrates superposition principle
- Shows an exponential dominant evolution
- Illustrates time system behavior

Vector Fields: Geometry of the System

- A system $\mathbf{x}' = A\mathbf{x}$ defines a **vector field**:

$$\mathbf{F}(\mathbf{x}) = A\mathbf{x}$$

- At each point $\mathbf{x} \in \mathbb{R}^2$, assign a vector $\mathbf{x}'$
- This field describes the **instantaneous direction** of motion

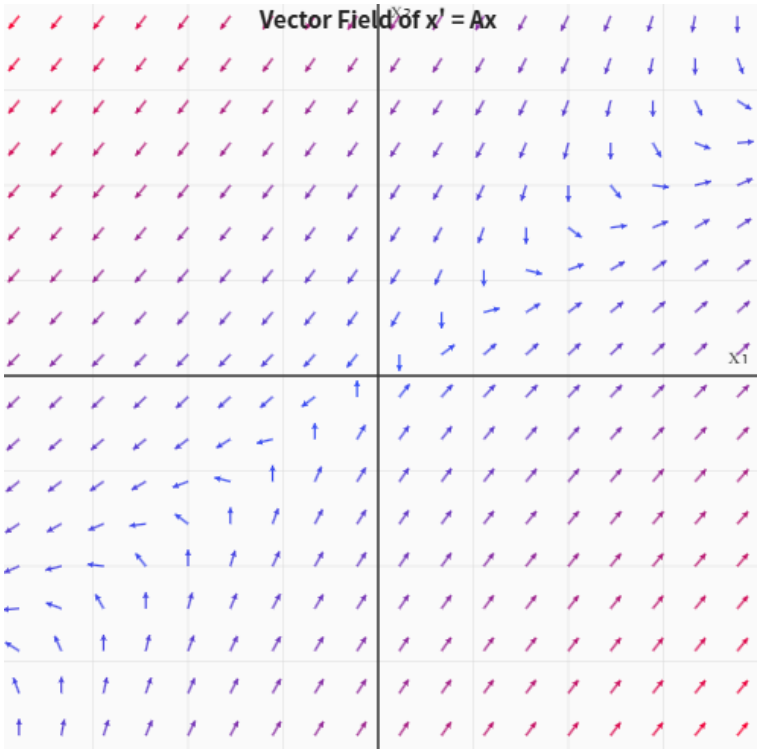


Figure: Visualizing it helps understand solution dynamics

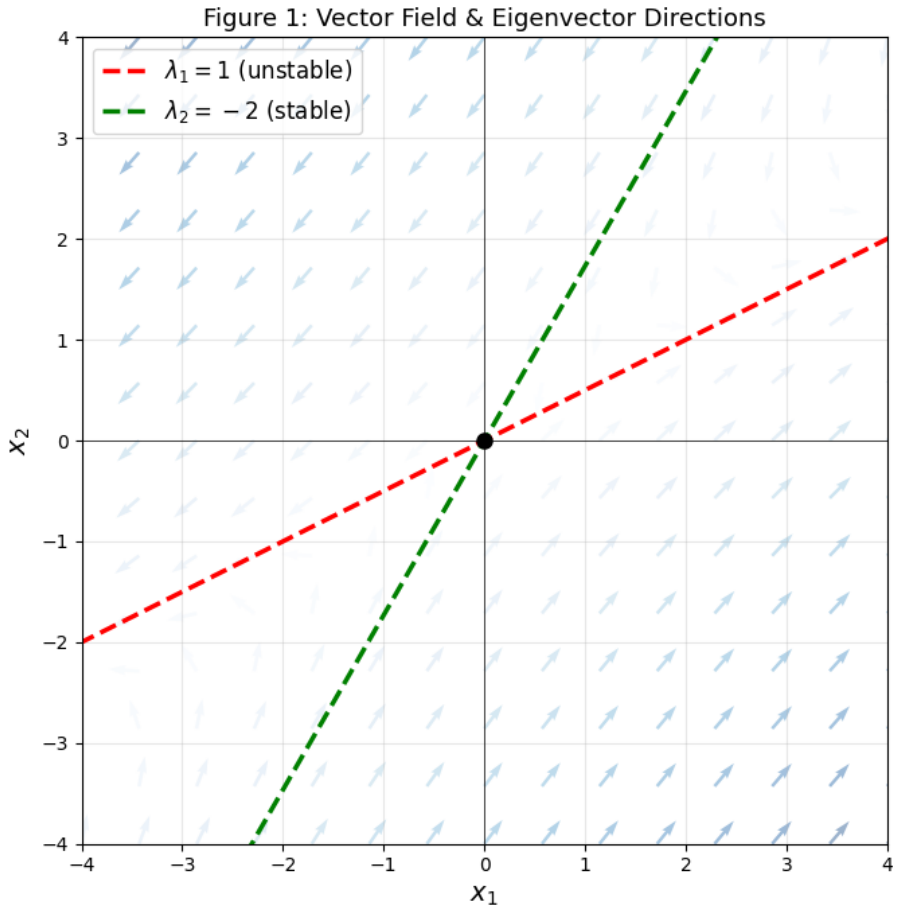
Phase Portrait (plain)

Key Features:

- Eigenvectors define solution directions
- $\lambda > 0$: Exponential growth along $\mathbf{v}_1$
- $\lambda < 0$: Exponential decay along $\mathbf{v}_2$

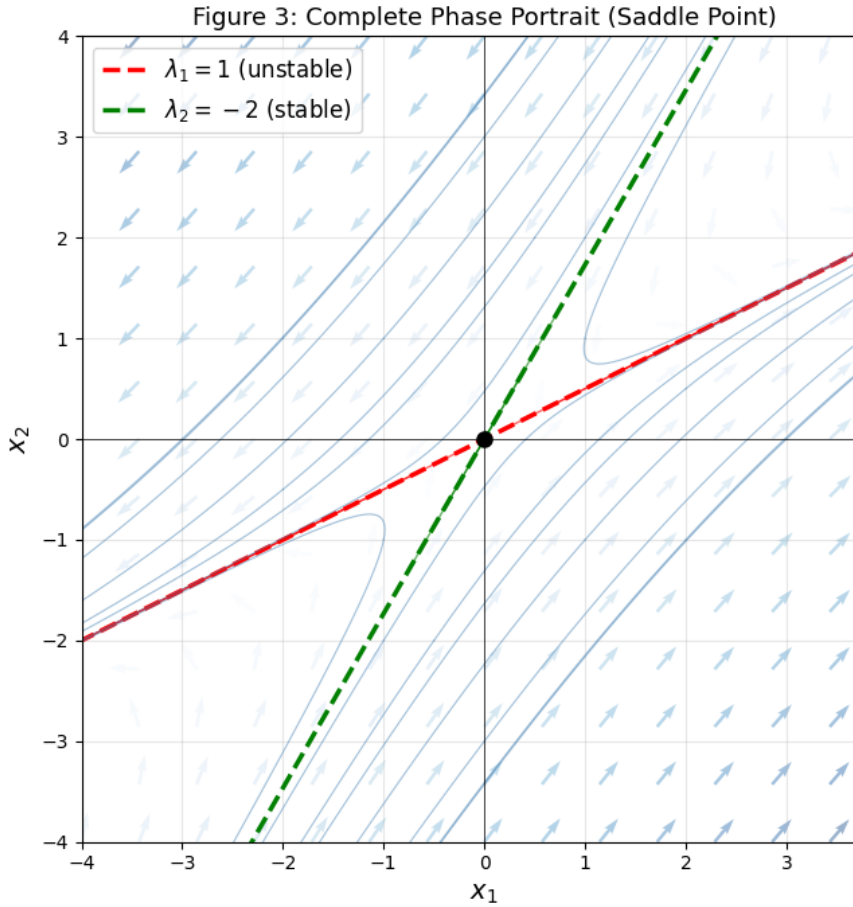
Classification:

- Saddle point: Eigenvalues with opposite signs
- Node: Real eigenvalues with same sign



Phase Portraits and Dynamics

- A **phase portrait** shows the trajectories of solutions in the phase plane
- **Eigenvectors** define invariant lines (solutions stay on them)
- If $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$:
  - $\lambda > 0$: solution diverges (unstable)
  - $\lambda < 0$: solution decays (stable)
- Classification of equilibrium at origin:
  - **Stable node**: two negative real eigenvalues
  - **Unstable node**: two positive real eigenvalues
  - **Saddle point**: eigenvalues with opposite signs
  - **Spiral / Center**: complex eigenvalues



From Geometry to Algebra

- Phase portraits show:
  - Straight-line or spiral solutions
  - Direction aligned with eigenvectors
  - Growth or decay rates driven by eigenvalues
- This geometric behavior matches the algebraic solution:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots$$

- **The vector field encodes the entire solution structure!**

Original Higher-Order ODE

Consider the third-order ODE:

$$y''' - 2y'' + y' - 3y = \sin(t)$$

with initial conditions:

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2$$

Our goal is to convert this into a system of first-order ODEs.

Define Substitutions

Let us define:

$$y_1 = y$$
$$y_2 = y'$$
$$y_3 = y''$$

Then their derivatives are:

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = \boxed{\sin(t) + 2y_3 - y_2 + 3y_1}$$

So we now have a system of first-order ODEs.

First-Order System with Initial Conditions

The first-order system is:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ \sin(t) + 2y_3 - y_2 + 3y_1 \end{bmatrix}$$

With initial conditions:

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 2$$

or $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, pls solve this ODE system with colab.

Model Examples

- SIR Infectious Disease Model
- Lotka-Volterra (Predator-Prey) Model

SIR Model: Introduction

- A compartmental model in epidemiology
- Population divided into three groups:
  - **S**usceptible: individuals who can contract the disease
  - **I**nfectious: individuals who have the disease and can transmit it
  - **R**ecovered: individuals who have recovered or died, and are no longer infectious
- Total population size: $N = S(t) + I(t) + R(t)$
- **Key parameters**:
  - β: transmission rate — the average number of contacts per person per time that are sufficient for transmission;
  - γ: recovery rate — the rate at which infectious individuals recover and move into the recovered population.

SIR Model: Structure

- The SIR model describes how a contagious disease spreads through a population over time.
- It is governed by a system of ordinary differential equations (ODEs),
- **The dynamics of the model are given by:**

$$\frac{dS}{dt} = -\beta \frac{SI}{N} \quad \text{(susceptible individuals become infected)}$$

$$\frac{dI}{dt} = \beta \frac{SI}{N} - \gamma I \quad \text{(infected individuals recover or continue spreading)}$$

$$\frac{dR}{dt} = \gamma I \quad \text{(infected individuals recover and gain immunity)}$$

where $N = S + I + R$ is the total population.

SIR Model: Assumptions

- Closed population (no births, deaths, or migration)
- Homogeneous mixing (each individual equally likely to meet any other)
- Constant transmission and recovery rates
- Instantaneous transition between compartments(no intermediate state for individuals)

SIR Model: Interpretation and Use

- **Basic Reproduction Number**:

$$R_0 = \frac{\beta}{\gamma}$$

- $R_0 > 1$: outbreak can spread in the population
- $R_0 < 1$: infection will die out
- Helps in understanding:
  - Epidemic threshold
  - Peak infection time and size
  - Impact of interventions like vaccination

SIR Model: Applications

- Applied to modeling outbreaks such as SARS, COVID-19, etc.
- Example: Hong Kong and Shanghai during an epidemic
- Model calibration using real-world data
- Comparison of intervention strategies between regions

Lotka-Volterra Model: Introduction

- A classic model in mathematical biology for predator-prey interactions.
- Describes the dynamics of biological systems in which two species interact:
  - **Prey** population (e.g., rabbits)
  - **Predator** population (e.g., foxes)
- The model assumes:
  - Prey population grows exponentially in absence of predators.
  - Predators rely entirely on prey for food.
  - Interaction between species affects population sizes.

Lotka-Volterra Model: Structure

- The model is defined by a system of two ordinary differential equations:

$$\frac{dx}{dt} = \alpha x - \beta xy \quad \text{(prey population)}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \quad \text{(predator population)}$$

- Where:
  - $x(t)$: number of prey at time t, and if $y(t) \equiv 0$, then $x(t) = e^{\alpha t}$
  - $y(t)$: number of predators at time t, and if $x(t) \equiv 0$, then y would be decreasing to a semi-positive constant, which is 0
  - α: natural growth rate of prey
  - β: predation rate coefficient
  - γ: natural death rate of predators
  - δ: rate at which consumed prey is converted into predators

Lotka-Volterra Model: Assumptions

- The environment does not change in ways that affect the interaction.
- Predators only survive by consuming prey.
- Prey have unlimited food supply (no intra-species competition).
- No immigration or emigration.
- Interaction between species is proportional to the product of their populations.

Lotka-Volterra Model: Interpretation

- The model produces oscillatory dynamics:
  - Prey population rises $\rightarrow$ predator population increases.
  - More predators $\rightarrow$ prey population declines.
  - Less prey $\rightarrow$ predator population declines.
  - Fewer predators $\rightarrow$ prey can recover.
- These cycles continue indefinitely under ideal conditions.
- Real ecosystems have additional complexities (e.g., carrying capacities, multiple species).

Lotka-Volterra Model: Applications

- Used to understand population dynamics in ecology.
- Applied to:
  - Predator-prey systems
  - Host-parasite interactions
  - Competition between species
- Provides insight into:
  - Sustainability of ecosystems
  - Effects of hunting, conservation, and environmental changes

Modeling: Role of Theory

- Mathematical models are abstractions of real systems guided by biological, physical, or social theory.
- Theory informs:
  - Which variables to include (e.g., compartments, species)
  - The form of interactions (e.g., linear vs. nonlinear terms)
  - The structure of equations (e.g., ODEs, difference equations)
- Example: SIR model based on assumptions about disease transmission and recovery.

Choosing Parameters: Theoretical and Empirical Sources

- Parameters can come from:
  - **Theory**: e.g., recovery time $\Rightarrow \gamma = \frac{1}{T}$
  - **Empirical data**: e.g., case counts, population surveys
  - **Experiments**: e.g., lab measurements of growth or infection rates
- Role of units and dimensional analysis in parameter consistency
- Sensitivity: small changes in parameters can lead to large changes in outcomes

Parameter Estimation: Example from SIR

- Suppose we observe an outbreak and gather time-series data on infected individuals.
- Use the reported average infectious period T to estimate $\gamma = 1/T$
- Estimate β by fitting the model to the data (e.g., least squares or MLE)
- **Example**:

$$\gamma = \frac{1}{7} \approx 0.143 \quad \text{(7-day infectious period)}$$

$$R_0 = 2.5 \Rightarrow \beta = R_0 \cdot \gamma = 2.5 \cdot 0.143 \approx 0.357$$

Impact of Assumptions and Simplifications

- Every model involves simplifying assumptions.
- Unrealistic assumptions can lead to misleading predictions.
- Examples:
  - Homogeneous mixing in SIR model may not hold in real populations.
  - Constant rates may ignore seasonality or behavior change.
- Sensitivity analysis and model validation are critical.

Next part: Numerical solutions

- Most nonlinear systems cannot be solved analytically, and when analysis is too difficult, we turn to numerical methods
- Idea: approximate solution step-by-step
- Widely used in engineering, physics, biology, finance, etc.

Why Numerical Solutions?

- Numerical methods approximate solutions given:
  - A differential equation: $\dot{x} = f(x, t)$
  - An initial condition: $x(t_0) = x_0$
- Main goals:
  - Accuracy
  - Stability
  - Efficiency
- Will also discuss practical implementation in colab with Python
- Basic methods: Euler, Runge-Kutta

Euler's Method

- Simplest numerical method
- Given $\dot{x} = f(x, t)$ and $x(t_0) = x_0$
- Step forward using:

$$x_{n+1} = x_n + hf(x_n, t_n)$$

- h is the time step (step size)
- Pros: easy to implement
- Cons: low accuracy, unstable for **stiff systems**.

Backward Euler: Method and Stability

- Implicit method:

$$x_{n+1} = x_n + hf(x_{n+1}, t_{n+1})$$

- Taylor expansion: global error $\mathcal{O}(h)$
- Stability analysis (test eq. $\dot{x} = \lambda x$):

$$x_{n+1} = x_n + h\lambda x_{n+1} \Rightarrow x_{n+1} = \frac{x_n}{1 - h\lambda}$$

- So $|x_{n+1}/x_n| = \left|\frac{1}{1-h\lambda}\right|$
- If $\mathrm{Re}(\lambda) < 0$, then method is **A-stable**

Runge-Kutta Methods (RK4)

- More accurate than Euler
- Classical 4th-order Runge-Kutta (RK4):

$$
\begin{aligned}
k_1 &= f(x_n, t_n) \\
k_2 &= f\left(x_n + \frac{h}{2}k_1, t_n + \frac{h}{2}\right) \\
k_3 &= f\left(x_n + \frac{h}{2}k_2, t_n + \frac{h}{2}\right) \\
k_4 &= f\left(x_n + hk_3, t_n + h\right) \\
x_{n+1} &= x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

- Widely used due to balance of accuracy and efficiency

Step Size, Error, and Stability

- Step size h affects:
  - Accuracy: smaller $h \Rightarrow$ more accurate
  - Computation time: smaller $h \Rightarrow$ more steps
- Error types:
  - Local truncation error (per step)
  - Global error (cumulative)
- Stability:
  - Some methods unstable for large h
  - Especially important for stiff equations

Forward Euler: Error and Stability

- Consider IVP: $\dot{x} = f(x, t)$, with $x(t_0) = x_0$
- Taylor expansion about t_n:

$$x(t_{n+1}) = x(t_n) + hf(x(t_n), t_n) + \frac{h^2}{2}x''(\xi), \quad \xi \in [t_n, t_{n+1}]$$

- Forward Euler uses:

$$x_{n+1} = x_n + hf(x_n, t_n)$$

- So local truncation error (LTE):

$$\text{LTE} = x(t_{n+1}) - x_{n+1} = \frac{h^2}{2}x''(\xi) \Rightarrow \mathcal{O}(h^2)$$

- Global error accumulates: $\mathcal{O}(h)$
- Stability analysis (test eq: $\dot{x} = \lambda x$):

$$x_{n+1} = (1 + h\lambda)x_n$$

- Method is stable if $|1 + h\lambda| < 1$
- For stiff problems (large negative λ), this requires very small h

Forward Euler: Error and Stability

- **Local truncation error (LTE)**: $\mathcal{O}(h^2)$
- **Global error**: $\mathcal{O}(h)$
- **Stability:**
  - Consider test equation: $\dot{x} = \lambda x$
  - Forward Euler: $x_{n+1} = x_n + h\lambda x_n = (1 + h\lambda)x_n$
  - Stable if $|1 + h\lambda| < 1$
  - For stiff problems (large negative λ), requires very small h
- **Conclusion:** Not suitable for stiff systems

RK4: Error and Stability

- RK4 uses intermediate slopes k_1, k_2, k_3, k_4
- Taylor expansion of exact solution gives:

$$x(t_{n+1}) = x(t_n) + hf + \frac{h^2}{2} f'f + \frac{h^3}{6} f''f^2 + \cdots$$

- RK4 matches up to 4th-order terms $\Rightarrow$ LTE: $\mathcal{O}(h^5)$
- Global error: $\mathcal{O}(h^4)$
- RK4 has a larger stability region than Forward Euler

Backward Euler Method

- **Implicit method**, good for stiff systems
- Given: $\dot{x} = f(x, t)$, then

$$x_{n+1} = x_n + hf(x_{n+1}, t_{n+1})$$

- Requires solving nonlinear equation at each step (e.g., via Newton's method)
- **Stability:**
  - For $\dot{x} = \lambda x$: $x_{n+1} = \frac{x_n}{1-h\lambda}$
  - Unconditionally stable if $\mathrm{Re}(\lambda) < 0$
- **Conclusion:** Preferred for stiff systems despite higher computational cost

Backward Euler: Solving the System Practically

If $\mathbf{f}$ is not complex, there are other ways to solve the implicit scheme, e.g. with inverse matrices

- For system: $\dot{\mathbf{x}} = f(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^n$
- Backward Euler update:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + hf(\mathbf{x}_{n+1}, t_{n+1})$$

- This is an implicit equation in $\mathbf{x}_{n+1}$
- To solve:
  - Apply Newton's method: iterate

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \left[I - hJ_f(\mathbf{x}^{(k)})\right]^{-1} \left[\mathbf{x}^{(k)} - \mathbf{x}_n - hf(\mathbf{x}^{(k)}, t_{n+1})\right]$$

  - Each iteration solves a linear system:

$$[I - hJ_f]\,\delta\mathbf{x} = -\left(\mathbf{x}^{(k)} - \mathbf{x}_n - hf(\mathbf{x}^{(k)}, t_{n+1})\right)$$

- This is a nonlinear system $\Rightarrow$ use iterative solvers at each time step

Example: Nonlinear 2D System

- Consider:

$$\begin{cases} \dot{x} = x(1-y) \\ \dot{y} = y(x-1) \end{cases}$$

- Forward Euler:

$$\begin{cases} x_{n+1} = x_n + h\, x_n(1-y_n) \\ y_{n+1} = y_n + h\, y_n(x_n-1) \end{cases}$$

- Runge-Kutta 4 (RK4):

$$\begin{aligned} k_1^x &= x_n(1-y_n), & k_1^y &= y_n(x_n-1) \\ k_2^x &= (x_n + \tfrac{h}{2}k_1^x)(1-(y_n+\tfrac{h}{2}k_1^y)), & k_2^y &= (y_n+\tfrac{h}{2}k_1^y)((x_n+\tfrac{h}{2}k_1^x)-1) \\ k_3^x &= (x_n + \tfrac{h}{2}k_2^x)(1-(y_n+\tfrac{h}{2}k_2^y)), & k_3^y &= (y_n+\tfrac{h}{2}k_2^y)((x_n+\tfrac{h}{2}k_2^x)-1) \\ k_4^x &= (x_n + h\,k_3^x)(1-(y_n+h\,k_3^y)), & k_4^y &= (y_n+h\,k_3^y)((x_n+h\,k_3^x)-1) \end{aligned}$$

$$\begin{cases} x_{n+1} = x_n + \frac{h}{6}(k_1^x + 2k_2^x + 2k_3^x + k_4^x) \\ y_{n+1} = y_n + \frac{h}{6}(k_1^y + 2k_2^y + 2k_3^y + k_4^y) \end{cases}$$

Summary and Next Steps

- Key theoretical concepts in ODE systems are complete.
- Now, we move on to applying the theory through practical computation.
- Next, we have prepared a session to practice everything we've learned using Colab and Python. You can refer back to these slides later as a guide when coding.

Python/Colab Practical Session

Will be covered in Colab notebook

Problem 1 — Qualitative Phase Portraits & Linearization

For each system below, complete parts (a)–(d):

(a) Find all equilibrium points.

(b) Write down the nullclines ($x' = 0$ and $y' = 0$). Use them to sketch a typical phase portrait.

(c) Compute the Jacobian matrix $J(x, y)$. For each equilibrium point, compute the eigenvalues.

(d) Classify each equilibrium point and state its stability.

System A:

$$\begin{cases} x' = x(1 - x) - xy, \\ y' = -y + xy. \end{cases}$$

System B:

$$\begin{cases} x' = x(1 - y), \\ y' = y(2 - x). \end{cases}$$

Problem 2 — Euler's Method for a System

Consider the linear system

$$\begin{cases} x' = -x + y, \\ y' = -2x - y, \end{cases} \qquad x(0) = 1,\ y(0) = 0.$$

With $\Delta t = 0.05$ and the explicit Euler method:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + 0.05 \begin{pmatrix} -x_n + y_n \\ -2x_n - y_n \end{pmatrix}$$

approximate the solution on $t = 0, 0.05, \ldots, 0.50$.

Tasks:

(a) Present a table of (t_n, x_n, y_n) for $n = 0, 1, \ldots, 10$.

(b) Plot the discrete trajectory in the xy-plane (connect the points and indicate the direction of increasing t).

(c) Describe whether the trajectory moves inward, outward, or circles around the origin. Justify by discussing $\sqrt{x_n^2 + y_n^2}$.