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HSMMC Pre-workshop Exercise (Linear Algebra) Solutions

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1. Compute $-2u + 4v$ where $u = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$.

   Answer: $-2u + 4v = -2\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 16 \\ 20 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \\ 16 \end{pmatrix}$

2. (a) Calculate $\begin{pmatrix} 3 & 7 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}$.

   (b) Calculate the inverse of $\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$.

   Answer:

   (a) $\begin{pmatrix} 3 & 7 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} (3)(-2)+(7)(4) & (3)(1)+7(2) \\ (-1)(-2)+(4)(4) & (-1)(1)+(4)(2) \end{pmatrix} = \begin{pmatrix} 22 & 17 \\ 18 & 7 \end{pmatrix}$

   (b) $\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}^{-1} = \frac{1}{(5)(4)-(3)(6)}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$

3. Define $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

   (a) Calculate A^2, A^3, A^4.

   (b) Hence, write down A^n (where n is a positive integer).

   (c) Similarly, find $(A^{-1})^n$.

   Answer:

   (a)

$$A^2 = AA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ (note: it can be proved by mathematical induction)

(c) $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$ (note: it can be proved by mathematical induction)

4. Please try the computational exercises in the Google Colab notebooks.

   Answer: See the solution notebook.

5. (Bonus) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $C \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for some non-zero vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and distinct scalars λ_1 and λ_2.

   (a) Prove that $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$.

   (b) Hence, prove that λ_1 and λ_2 are the roots of the equation $\lambda^2 - tr(C) \cdot \lambda + \det(C) = 0$. (Here, $tr(C)$ = the sum of all diagonal entries of $C = p + s$, which is also known as the *trace* of a matrix.)

   Answer:

   (a) Since $C \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for some non-zero vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, we have $(C - \lambda_1 I) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, i.e.,

   $$\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

   If $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} \neq 0$, then the matrix inverse $\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix}^{-1}$ exists. Multiplying both sides of the above equation by $\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix}^{-1}$ gives $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, which contradicts the condition that the vector is a non-zero vector. Therefore, we have $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$.

   Similarly, $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$.

(b) From (a), we have

$$\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = (p-\lambda_1)(s-\lambda_1) - rq = 0,$$

from which we have

$$\lambda_1^2 - (p+s)\lambda_1 + ps - rq = 0.$$

Similarly,

$$\lambda_2^2 - (p+s)\lambda_2 + ps - rq = 0.$$

Also, note that $p+s = tr(C)$ and $ps - rq = \det(C)$. Therefore, λ_1, λ_2 are the two roots of the solution

$$\lambda^2 - tr(C) \cdot \lambda + \det(C) = 0.$$

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1. 計算 $-2u+4v$，其中 $u=\begin{pmatrix}-2\\-2\\2\end{pmatrix}$，$v=\begin{pmatrix}0\\4\\5\end{pmatrix}$。

答案：$-2u+4v=-2\begin{pmatrix}-2\\-2\\2\end{pmatrix}+4\begin{pmatrix}0\\4\\5\end{pmatrix}=\begin{pmatrix}4\\4\\-4\end{pmatrix}+\begin{pmatrix}0\\16\\20\end{pmatrix}=\begin{pmatrix}4\\20\\16\end{pmatrix}$

2. (a) 計算 $\begin{pmatrix}3&7\\-1&4\end{pmatrix}\begin{pmatrix}-2&1\\4&2\end{pmatrix}$。

(b) 計算 $\begin{pmatrix}5&3\\6&4\end{pmatrix}$ 的逆矩陣。

答案：

(a) $\begin{pmatrix}3&7\\-1&4\end{pmatrix}\begin{pmatrix}-2&1\\4&2\end{pmatrix}=\begin{pmatrix}(3)(-2)+(7)(4)&(3)(1)+7(2)\\(-1)(-2)+(4)(4)&(-1)(1)+(4)(2)\end{pmatrix}=\begin{pmatrix}22&17\\18&7\end{pmatrix}$

(b) $\begin{pmatrix}5&3\\6&4\end{pmatrix}^{-1}=\frac{1}{(5)(4)-(3)(6)}\begin{pmatrix}4&-3\\-6&5\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4&-3\\-6&5\end{pmatrix}$

3. 設 $A=\begin{pmatrix}1&0\\1&1\end{pmatrix}$。

(a) 計算 A^2，A^3，A^4。

(b) 由此，歸納出 A^n（其中 n 為正整數）。

(c) 同理，求 $(A^{-1})^n$。

答案：

(a)

$$A^2=AA=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\2&1\end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$（可用數學歸納法證明）

(c) $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$（可用數學歸納法證明）

4. 嘗試完成 Google Colab 筆記本中的計算練習。

答案：請參閱習題解答。

5. （挑戰題）設 $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$。已知存在非零向量 $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$、$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ 及相異純量 λ_1 和 λ_2，使得 $C\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ 和 $C\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$。

(a) 證明 $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$ 及 $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$。

(b) 由此，證明 λ_1 和 λ_2 為方程 $\lambda^2 - \mathrm{tr}(C)\cdot\lambda + \det(C) = 0$ 的根。
（其中 $\mathrm{tr}(C) = C$ 的對角線元素之和 $= p + s$，亦稱為矩陣的跡。）

答案：

(a) 由於存在非零向量 $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ 使得 $C\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$，可得 $(C - \lambda_1 I)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$，即

$$\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

若 $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} \neq 0$，則逆矩陣 $\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix}^{-1}$ 存在。將上式兩邊同時左乘 $\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix}^{-1}$ 可得 $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$，這與該向量為非零向量的條件矛盾。因此，我們有 $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$。同理可證 $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$。

(b) 由 (a) 可得

$$\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = (p-\lambda_1)(s-\lambda_1) - rq = 0,$$

整理後得

$$\lambda_1^2-(p+s)\lambda_1+ps-rq=0.$$

同樣地，

$$\lambda_2^2-(p+s)\lambda_2+ps-rq=0.$$

注意到 $p+s=\mathrm{tr}(C)$ 及 $ps-rq=\det(C)$。因此，λ_1,λ_2 為方程

$$\lambda^2-\mathrm{tr}(C)\cdot\lambda+\det(C)=0$$

的兩個根。