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HSMMC Pre-workshop Learning Materials (Differential Equations)

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Background 背景

Many real-world problems can be described or modelled by differential equations, such as:
許多現實世界的問題都可以用微分方程來描述或建模，例如：

- Laws of motions 運動定律
- Heat diffusion 熱擴散
- Biological growth 生物增長
- ...

Here, we cover some basic concepts of differential equations and techniques for solving differential equations.
在此，我們涵蓋一些微分方程的基本概念以及求解微分方程的技巧。

1 Differential Equations 微分方程

A **differential equation** is an equation containing derivatives as an unknown function. The **order** of a differential equation is the order of the highest derivative in the equation.
微分方程是一個包含未知函數導數的方程。微分方程的**階數**是方程中最高階導數的階數。

Example 例子

Differential Equation	Order	Unknown Function
$\frac{dy}{dx} = 4y$	1	$y(x)$
$y'' + 2y = 2x$	2	$y(x)$
$\frac{d^3y}{dt^3} - t\frac{dy}{dt} + t(y-1) = e^t$	3	$y(t)$

In some cases, we can solve a differential equation by directly integrating the equation on both sides.
在某些情況下，我們可以通過對方程兩邊直接積分來求解微分方程。

Example 例子

Consider the differential equation
考慮微分方程

$$y' = x.$$

To solve it, we integrate the equation with respect to x once,
為求解它，我們對 x 積分一次，

$$y = \int x dx = \frac{1}{2}x^2 + C,$$

where $C \in \mathbb{R}$. This is called the **general solution**.
其中 $C \in \mathbb{R}$。這稱為**通解**。

If the integration constant C is set to be a particular number, e.g. $C = 3$, then $y = \frac{1}{2}x^2 + 3$ is a **particular solution** to the differential equation.
若積分常數 C 取特定數值，例如 $C = 3$，則 $y = \frac{1}{2}x^2 + 3$ 是該微分方程的一個**特解**。

The general solution to a differential equation contains an arbitrary constant. To determine a unique solution, some conditions need to be described. For example, consider
微分方程的通解包含一個任意常數。為確定唯一解，需要給定一些條件。例如，考慮

$$\begin{cases} y' = x, \\ y(0) = 3. \end{cases}$$

The general solutions are $y = \frac{1}{2}x^2 + C$. By using the **initial condition** $y(0) = 3$, we get $C = 3$. This problem is called an **initial condition problem**.
通解為 $y = \frac{1}{2}x^2 + C$。利用**初始條件** $y(0) = 3$，我們得到 $C = 3$。此問題稱為**初值問題**。

2 Separable Differential Equation 可分離微分方程

Differential equations in the following form
下列形式的微分方程

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

are called **separable**.
稱為**可分離**的。

To solve the above equation, we rewrite it as
為求解上述方程，我們將其改寫為

$$h(y)\frac{dy}{dx} = g(x)$$

Integrate with respect to x,
對 x 積分，

$$\int h(y)dy = \int g(x)dx,$$
$$H(y) = G(x) + C,$$

which determines the function (relation) y about x.
這確定了 y 關於 x 的函數（關係）。

It is more convenient to denote the equation by
更便捷的記法是將方程寫成

$$h(y)dy = g(x)dx,$$

and the name **"separable"** just means the variables x and y are separated on two sides.
而「**可分離**」這一名稱正意味著變量 x 和 y 被分離在兩側。

Example 例子

Solve the differential equation
求解微分方程

$$\frac{dy}{dx} = \frac{2x^3}{y^2}.$$

Then solve the initial value problem
然後求解初值問題

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2x^3}{y^2}, \\ y(0) = 1. \end{cases}$$

Solution. The equation is rewritten as
解： 將方程改寫為

$$y^2 dy = 2x^3 dx.$$

Integrate both sides,
對兩邊積分，

$$\int y^2 dy = \int 2x^3 dx,$$

Hence the general solution is
因此通解為

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + C.$$

Substituting the initial condition $y(0) = 1$ into the general solution yields $C = \frac{1}{3}$. Thus, the solution to the initial value problem is
將初始條件 $y(0) = 1$ 代入通解，得 $C = \frac{1}{3}$。因此，該初值問題的解為

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + \frac{1}{3}.$$

3 First-Order Linear Differential Equation 一階線性微分方程

Equations in the following form
下列形式的方程

$$\frac{dy}{dx} + p(x)y = q(x)$$

are called **first-order linear differential equations**.
稱為**一階線性微分方程**。

If $p(x)=0$, integrate the equation $\frac{dy}{dx}=q(x)$ to get
若 $p(x)=0$，對方程 $\frac{dy}{dx}=q(x)$ 積分可得

$$y=\int q(x)dx.$$

If $p(x)\neq 0$, multiply the equation by some function
若 $p(x)\neq 0$，將方程乘以某個函數 $\mu=\mu(x)$,

$$\mu\frac{dy}{dx}+\mu p(x)y=\mu q(x). \tag{1}$$

This can be done as long as μ satisfies
這只要 μ 滿足以下條件就可以完成：

$$\frac{d\mu}{dx}=\mu p(x),$$

which is a separable equation so that the left-hand side is a derivative of μy, that is,
這是一個可分離方程，使得左邊成為 μy 的導數，即

$$\mu\frac{dy}{dx}+\mu p(x)y=\frac{d}{dx}(\mu y)=\mu\frac{dy}{dx}+\frac{d\mu}{dx}y.$$

This can be done as long as μ satisfies
這只要 μ 滿足以下條件就可以完成：

$$\frac{d\mu}{dx}=\mu p(x),$$

which is a separable equation whose general solution is given by
這是一個可分離方程，其通解為

$$\mu=e^{\int p(x)dx+\tilde{C}}. \tag{2}$$

(Note: μ is called the *integrating factor* and plays an important role in the general solution of the original first-order differential equation as shown below. It can be proved that the constant $\tilde{C}$ here will not affect the final general solution of the original differential equation. Therefore, it is common to directly set $\tilde{C}=0$ and take $\mu=e^{\int p(x)dx}$.)
（註：μ 稱為積分因子，在求解原一階微分方程的通解中起重要作用，如下所示。可以證明此處的常數 $\tilde{C}$ 不會影響原微分方程的最終通解。因此，我們通常直接設 $\tilde{C}=0$ 並取 $\mu=e^{\int p(x)dx}$。）

Returning to (1), we have
回到 (1)，我們有

$$\frac{d}{dx}(\mu y)=\mu q(x)$$

Hence
因此

$$\mu y=\int \mu q(x)dx+C$$

$$y=\frac{1}{\mu}(\int \mu q(x)dx+C) \tag{3}$$

Example 例子

Solve

求解

$$\frac{dy}{dx} - y = e^{3x}.$$

Solution. It is a first order linear equation with $p(x) = -1$ and $q(x) = e^{3x}$. Let

解： 這是一階線性方程，其中 $p(x) = -1$，$q(x) = e^{3x}$。設

$$\mu = e^{\int p(x)dx} = e^{-x}.$$

(Caution: it should be e^{-x+C}, but we choose $C = 0$.)

注意：應為 e^{-x+C}，但我們取 $C = 0$。）

Multiply the equation by μ,

將方程乘以 μ，

$$\frac{d}{dx}(e^{-x}y) = e^{-x}e^{3x} = e^{2x}.$$

So

因此

$$e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

(Caution: this C cannot be omitted.)

（注意：此處 C 不可省略。）

Hence, the general solution is

因此，通解為

$$y = e^{x}(\frac{1}{2}e^{2x} + C).$$

Further readings
延伸閱讀

- Differential Equations self-learning materials on Khan Academy:
  Khan Academy的微分方程自學材料：
  `https://www.khanacademy.org/math/differential-equations`

- Differential Equations self-learning materials on MIT OpenCourseWare:
  MIT OpenCourseWare 的微分方程自學材料：
  `https://ocw.mit.edu/courses/18-03-differential-equations-spring-2010`