

THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

Mathematical Modelling Project Team

mathmodel@math.cuhk.edu.hk

HSMMC Pre-workshop Exercise (Differential Equations) Solutions

Last updated: March 23, 2026

1. Solve the following separable differential equations.

(a) $\frac{dy}{dx} = -4x^2y^3$;

(b) $\frac{dy}{dx} = \frac{y^2 + 1}{xy}$ with $y(1) = 3$;

(c) $\frac{dy}{dx} = xe^{y-x^2}$ with $y(1) = 0$.

Answer:

(a) If $y = 0$, it is easy to see that the differential equation holds.

If $y \neq 0$, we have

$$\frac{1}{y^3}dy = -4x^2dx$$

By integrating both sides, we have

$$\int \frac{1}{y^3}dy = \int -4x^2dx$$

$$\frac{1}{-2y^2} = \frac{-4x^3}{3} + C$$

$$\frac{1}{y^2} = \frac{8x^3 + C}{3}$$

$$y = \pm \sqrt{\frac{3}{8x^3 + C}}$$

Therefore, the solutions are $y = 0$ or $y = \pm \sqrt{\frac{3}{8x^3 + C}}$, where C is a constant.

(b) From the given differential equation, we have

$$\frac{y}{y^2 + 1}dy = \frac{1}{x}dx$$

By integrating both sides, we have

$$\int \frac{y}{y^2 + 1}dy = \int \frac{1}{x}dx$$

$$\int \frac{1}{2} \cdot \frac{2y}{y^2 + 1} dy = \int \frac{1}{x} dx$$

$$\frac{1}{2} \int \frac{1}{y^2 + 1} d(y^2) = \int \frac{1}{x} dx$$

$$\frac{1}{2} \ln |y^2 + 1| = \ln |x| + C$$

$$\ln(y^2 + 1) = \ln(x^2) + C$$

$$y^2 + 1 = e^{\ln(x^2) + C}$$

$$y = \pm \sqrt{e^C x^2 - 1}$$

Using the given condition $y(1) = 3$, we have

$$9 = e^C - 1 \Rightarrow e^C = 10$$

Therefore, the solution is

$$y = \sqrt{10x^2 - 1}.$$

(c) From the given differential equation, we have

$$e^{-y} dy = x e^{-x^2} dx$$

By integrating both sides,

$$\int e^{-y} dy = \int x e^{-x^2} dx$$

$$\int -e^{-y} dy = \int -x e^{-x^2} dx$$

$$\int -e^{-y} dy = \frac{1}{2} \int (-2x) e^{-x^2} dx$$

$$e^{-y} = \frac{1}{2} e^{-x^2} + C$$

Since $y(1) = 0$, we have $e^{-0} = \frac{1}{2} e^{-1^2} + C \Rightarrow C = 1 - \frac{1}{2e}$.

Hence, we have

$$e^{-y} = \frac{1}{2} e^{-x^2} + 1 - \frac{1}{2e}$$

$$y = -\ln \left(\frac{1}{2} e^{-x^2} + 1 - \frac{1}{2e} \right)$$

2. Solve the first-order linear differential equation

$$y' + y = x e^{-x} + 1.$$

Answer: The integrating factor is $\mu(x) = e^{\int 1 dx} = e^x$. Therefore, we have

$$e^x (y' + y) = e^x (x e^{-x} + 1)$$

$$(e^x y)' = x + e^x$$

$$e^x y = \frac{x^2}{2} + e^x + C$$
$$y = \frac{x^2}{2} e^{-x} + 1 + C e^{-x}$$

3. (Bonus) Consider the first-order linear differential equation

$$P(x)y' + Q(x)y = 0,$$

where $P(x)$, $Q(x)$ are given functions. Prove that if $y_1(x)$ and $y_2(x)$ are two solutions to the above equation, then for any constants c_1 and c_2 , the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution.

Answer: Substituting $y = c_1 y_1(x) + c_2 y_2(x)$ into the equation, we have

$$\begin{aligned} & P(x)y' + Q(x)y \\ &= P(x)(c_1 y_1 + c_2 y_2)' + Q(x)(c_1 y_1 + c_2 y_2) \\ &= P(x)(c_1 y_1' + c_2 y_2') + Q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1(P(x)y_1' + Q(x)y_1) + c_2(P(x)y_2' + Q(x)y_2) \\ &= 0 \end{aligned}$$

Therefore, $y = c_1 y_1(x) + c_2 y_2(x)$ is also a solution.

香港中文大學
數學系

數學建模計劃團隊
mathmodel@math.cuhk.edu.hk

香港－上海中學生數學建模比賽 (HSMBC) 工作坊前練習 (微分方程) 參考答案

最後更新: 2026年3月23日

1. 求解下列可分離微分方程。

(a) $\frac{dy}{dx} = -4x^2y^3$;

(b) $\frac{dy}{dx} = \frac{y^2 + 1}{xy}$, 其中 $y(1) = 3$;

(c) $\frac{dy}{dx} = xe^{y-x^2}$, 其中 $y(1) = 0$ 。

答案 :

(a) 若 $y = 0$, 很容易看出微分方程成立。

若 $y \neq 0$, 我們有

$$\frac{1}{y^3}dy = -4x^2dx$$

對兩邊積分, 得到

$$\int \frac{1}{y^3}dy = \int -4x^2dx$$

$$\frac{1}{-2y^2} = \frac{-4x^3}{3} + C$$

$$\frac{1}{y^2} = \frac{8x^3 + C}{3}$$

$$y = \pm \sqrt{\frac{3}{8x^3 + C}}$$

因此, 解為 $y = 0$ 或 $y = \pm \sqrt{\frac{3}{8x^3 + C}}$, 其中 C 為常數。

(b) 由給定的微分方程, 我們有

$$\frac{y}{y^2 + 1}dy = \frac{1}{x}dx$$

對兩邊積分, 得到

$$\int \frac{y}{y^2 + 1}dy = \int \frac{1}{x}dx$$

$$\int \frac{1}{2} \cdot \frac{2y}{y^2 + 1}dy = \int \frac{1}{x}dx$$

$$\begin{aligned} \frac{1}{2} \int \frac{1}{y^2 + 1} d(y^2) &= \int \frac{1}{x} dx \\ \frac{1}{2} \ln |y^2 + 1| &= \ln |x| + C \\ \ln(y^2 + 1) &= \ln(x^2) + C \\ y^2 + 1 &= e^{\ln(x^2) + C} \\ y &= \pm \sqrt{e^C x^2 - 1} \end{aligned}$$

利用給定條件 $y(1) = 3$ ，得到

$$9 = e^C - 1 \Rightarrow e^C = 10$$

因此，解為

$$y = \sqrt{10x^2 - 1}.$$

(c) 由給定的微分方程，我們有

$$e^{-y} dy = x e^{-x^2} dx$$

對兩邊積分，

$$\begin{aligned} \int e^{-y} dy &= \int x e^{-x^2} dx \\ \int -e^{-y} dy &= \int -x e^{-x^2} dx \\ \int -e^{-y} dy &= \frac{1}{2} \int (-2x) e^{-x^2} dx \\ e^{-y} &= \frac{1}{2} e^{-x^2} + C \end{aligned}$$

由於 $y(1) = 0$ ，我們有 $e^{-0} = \frac{1}{2} e^{-1^2} + C \Rightarrow C = 1 - \frac{1}{2e}$ 。
因此，

$$\begin{aligned} e^{-y} &= \frac{1}{2} e^{-x^2} + 1 - \frac{1}{2e} \\ y &= -\ln \left(\frac{1}{2} e^{-x^2} + 1 - \frac{1}{2e} \right) \end{aligned}$$

2. 求解一階線性微分方程

$$y' + y = x e^{-x} + 1.$$

答案：積分因子為 $\mu(x) = e^{\int 1 dx} = e^x$ 。因此，我們有

$$\begin{aligned} e^x (y' + y) &= e^x (x e^{-x} + 1) \\ (e^x y)' &= x + e^x \\ e^x y &= \frac{x^2}{2} + e^x + C \\ y &= \frac{x^2}{2} e^{-x} + 1 + C e^{-x} \end{aligned}$$

3. (挑戰題) 考慮一階線性微分方程

$$P(x)y' + Q(x)y = 0,$$

其中 $P(x)$ 、 $Q(x)$ 為已知函數。證明：若 $y_1(x)$ 和 $y_2(x)$ 是上述方程的兩個解，則對任意常數 c_1 和 c_2 ，函數

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

亦是解。

答案：將 $y = c_1y_1(x) + c_2y_2(x)$ 代入方程，我們有

$$\begin{aligned} & P(x)y' + Q(x)y \\ &= P(x)(c_1y_1 + c_2y_2)' + Q(x)(c_1y_1 + c_2y_2) \\ &= P(x)(c_1y_1' + c_2y_2') + Q(x)(c_1y_1 + c_2y_2) \\ &= c_1(P(x)y_1' + Q(x)y_1) + c_2(P(x)y_2' + Q(x)y_2) \\ &= 0 \end{aligned}$$

因此， $y = c_1y_1(x) + c_2y_2(x)$ 亦是解。