THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

Mathematical Modelling Project Team
mathmodel@math.cuhk.edu.hk

HSMMC Pre-workshop Exercise (Probability and Statistics) Solutions

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1. The probability that John passes a Math exam is $\frac{4}{5}$ and that he passes a Chemistry exam is $\frac{5}{6}$. If the probability that he passes both exams is $\frac{3}{4}$, find the probability that he will pass at least one exam.

   Answer: Let M = John passes the math exam, and C = John passes the chemistry exam. $P(\text{John passes at least one exam}) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{4}{5} + \frac{5}{6} - \frac{3}{4} = \frac{53}{60}$

2. A package of 6 light bulbs contains 2 defective bulbs. If 3 bulbs are selected for use, find the probability that none of the three is defective.

   Answer:

$$P(\text{none is defective}) = \frac{\text{number of ways that 3 non defectives can be chosen}}{\text{total number of ways that a sample of 3 can be chosen}} = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$$

3. Three bits (0 or 1 digits) are transmitted over a noisy channel, so they will be flipped independently with probability 0.1 each. What is the probability that

   (a) At least one bit is flipped?

   (b) Exactly one bit is flipped?

   Answer:

   (a) Using the complement rule, $P(\text{at least one}) = 1 - P(\text{none})$. If we denote F_k as the event that the kth bit is flipped, due to independence, we have

$$\begin{aligned} P(\text{at least one bit is flipped}) &= 1 - P(\text{no bits is flipped}) \\ &= 1 - P(F_1' \cap F_2' \cap F_3') \\ &= 1 - (1 - 0.1)^3 \\ &= 0.271 \end{aligned}$$

   (b) Flipping exactly one bit can be accomplished in 3 ways:

$$P(\text{exactly one}) = P(F_1 \cap F_2' \cap F_3') + P(F_1' \cap F_2 \cap F_3') + P(F_1' \cap F_2' \cap F_3) = 0.243$$

4. It is known that:

- The probability of having a rainy day is 0.1.
- The probability that a person arrives at work late is 0.2.
- The probability that the person arrives at work late given that the day is a rainy day is 0.8.

What is the probability that the day is a rainy day, given that the person arrives at work late?

Answer: By Bayes' Rule,

$$P(\text{rainy}|\text{late}) = \frac{P(\text{rainy})P(\text{late}|\text{rainy})}{P(\text{late})} = \frac{0.1 \times 0.8}{0.2} = 0.4$$

5. Let X be a random variable having a probability mass function given in the following.

x	0	1	2	3	4
$P(X = x)$	0.52	0.28	0.14	0.04	0.02

Calculate the mean and variance of the random variable $Y = 4X + 3$.

Answer: It is easy to find that $\mathbb{E}(X) = 0.76 = \mu$ and $V(X) = 1.52 - (0.76)^2 = 0.9424$
Then,

$$\mathbb{E}(Y) = \mathbb{E}(4X + 3) = \sum(4x + 3)p(x) = 4\sum xp(x) + 3\sum p(x) = 4\mathbb{E}(X) + 3 = 6.04$$

$$V(Y) = V(4X + 3) = V(4X) = 4^2V(X) = 15.08$$

6. (Bonus) Suppose that the error in the reaction temperature, in $°C$, for a controlled laboratory experiment is a continuous random variable X having the density

$$f(x) = \begin{cases} \frac{x^2}{3} & \text{for } -1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Verify that the probability indeed adds up to 1.

(b) Find $P(0 < X < 1)$.

Answer:

(a)

$$\int_{\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3}dx = \frac{8}{9} + \frac{1}{9} = 1$$

(b)

$$P(0 < X < 1) = \int_{0}^{1} \frac{x^2}{3}dx = \frac{1}{9}$$

香港中文大學
數學系

數學建模計劃團隊
mathmodel@math.cuhk.edu.hk

香港－上海中學生數學建模比賽 (HSMMC) 工作坊前練習（概率與統計）參考答案

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1. 約翰通過數學考試的概率為 $\frac{4}{5}$，通過化學考試的概率為 $\frac{5}{6}$。若他同時通過兩科考試的概率為 $\frac{3}{4}$，求他至少通過一科考試的概率。

   答案： 設 $M =$ 約翰通過數學考試，$C =$ 約翰通過化學考試。
   $P(\text{約翰至少通過一科考試}) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{4}{5} + \frac{5}{6} - \frac{3}{4} = \frac{53}{60}$

2. 一包 6 個的燈泡中含有 2 個壞的。若隨機抽取 3 個燈泡使用，求選出的 3 個燈泡均為正常（非壞）的概率。

   答案：

$$P(\text{三個均為正常}) = \frac{\text{選出 3 個正常燈泡的方法數}}{\text{選出 3 個燈泡的總方法數}} = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$$

3. 三個比特（0 或 1 數字）在一個有雜訊的通道中傳輸，每個比特獨立翻轉的概率均為 0.1。求下列概率：

   (a) 至少有一個比特被翻轉；

   (b) 恰好有一個比特被翻轉。

   答案：

   (a) 利用餘集法則，$P(\text{至少一個比特被翻轉}) = 1 - P(\text{沒有比特被翻轉})$。記 F_k 為第 k 個比特被翻轉的事件，由比特翻轉的獨立性，可得

$$\begin{aligned} P(\text{至少有一個比特被翻轉}) &= 1 - P(\text{沒有比特被翻轉}) \\ &= 1 - P(F_1' \cap F_2' \cap F_3') \\ &= 1 - (1 - 0.1)^3 \\ &= 0.271 \end{aligned}$$

   (b) 恰好翻轉一個比特可有 3 種方式：

$$P(\text{恰好一個}) = P(F_1 \cap F_2' \cap F_3') + P(F_1' \cap F_2 \cap F_3') + P(F_1' \cap F_2' \cap F_3) = 0.243$$

4. 已知：

- 下雨天的概率為 0.1。
- 某人上班遲到的概率為 0.2。
- 在雨天該人上班遲到的概率為 0.8。

求在該人上班遲到的情況下，當天為下雨天的概率。

答案：由貝氏定理，

$$P(\text{雨天}|\text{遲到}) = \frac{P(\text{雨天})P(\text{遲到}|\text{雨天})}{P(\text{遲到})} = \frac{0.1 \times 0.8}{0.2} = 0.4$$

5. 設 X 為一隨機變量，其概率質量函數如下表所示：

x	0	1	2	3	4
$P(X=x)$	0.52	0.28	0.14	0.04	0.02

計算隨機變量 $Y = 4X + 3$ 的期望值與方差。

答案： 不難發現 $\mathbb{E}(X) = 0.76 = \mu$，$V(X) = 1.52 - (0.76)^2 = 0.9424$。
則

$$\mathbb{E}(Y) = \mathbb{E}(4X+3) = \sum (4x+3)p(x) = 4\sum xp(x) + 3\sum p(x) = 4\mathbb{E}(X) + 3 = 6.04$$

$$V(Y) = V(4X+3) = V(4X) = 4^2 V(X) = 15.08$$

6. （挑戰題）設某對照實驗中反應溫度的誤差（以 $^{\circ}C$ 為單位）為一連續隨機變量 X，其密度函數為

$$f(x) = \begin{cases} \frac{x^2}{3} & \text{當 } -1 \le x \le 2 \\ 0 & \text{其他情況} \end{cases}$$

(a) 驗證該概率密度函數的總概率確實為 1。

(b) 求 $P(0 < X < 1)$。

答案：

(a)

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3}dx = \frac{8}{9} + \frac{1}{9} = 1$$

(b)

$$P(0 < X < 1) = \int_0^1 \frac{x^2}{3}dx = \frac{1}{9}$$