HSMMC Training Workshop 2
Advanced Calculus, Optimization, and Differential Equations

HSMMC 訓練工作坊 2
進階微積分、最優化及微分方程

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Last time: Advanced linear algebra and linear programming 進階線性代數及線性規劃

- Topics:
  - Linear Algebra Review, Least-Squares Problems, and Regularized Regression 線性代數複習、最小平方法問題及正則化迴歸
  - Eigenvalues and eigenvectors 特徵值和特徵向量
  - Graph theory and graph algorithms 圖論和圖算法
  - Linear programming 線性規劃
  - Integer programming 整數規劃
  - Network flow problem 網路流問題
- **Theory + computation (Python)**
  理論 + Python 計算

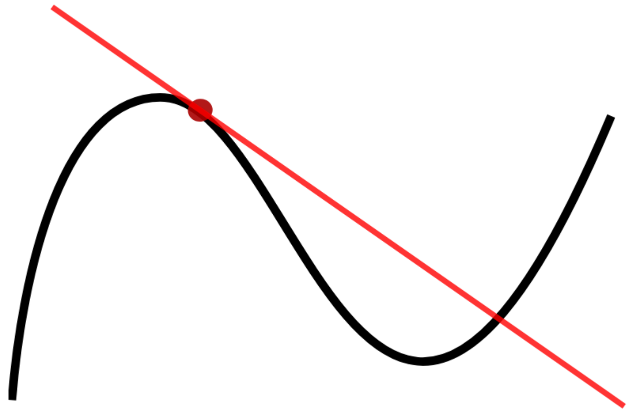
Today: Calculus, optimization, and differential equations
微積分、最優化及微分方程

- Differentiation 微分
- Integration 積分
- Optimization 最優化
- Differential equations 微分方程
- Systems of differential equations 微分方程組
- Goal:
  - Understanding the principle of optimization and differential equations
    理解優化與微分方程的原理
  - Utilizing IT tools (Python) to solve optimization and differential equation problems
    使用 Python 工具求解優化與微分方程問題
  - Modelling using optimization and differential equations
    學習如何用優化與微分方程進行實際建模

Differentiation 微分

- **Derivative**: rate of change of a function $y = f(x)$
  導數: 函數$y = f(x)$ 的變化率

$$\frac{dy}{dx} = y' = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



e.g. For $y = f(x) = x^2$, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x$$

- Applications 應用:
  - Modelling changes 建模變化
  - Finding maximum or minimum
    求函數的最大值或最小值

Differentiation rules and results 微分法則及結果

Derivatives of common functions 常見函數的導數:

- $\frac{d}{dx}(const) = 0$
- $\frac{d}{dx}x^a = ax^{a-1}$
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}\ln(x) = \frac{1}{x}$ for $x > 0$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x = \frac{1}{\cos^2 x}$

Differentiation rules and results 微分法則及結果

Differentiation rules 微分法則:

- $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$
- $\frac{d}{dx}(ku)=k\frac{du}{dx}$
- $\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{dv}{dx}$
  (product rule 乘積法則)
- $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-u\,v'}{v^2}$
  (quotient rule 商法則)
- $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ (chain rule 鏈鎖法則)

- $\left(x^2+x\right)'=2x+1$
- $\left(3x^2\right)'=3\left(x^2\right)'=6x$
- $(x\sin x)'=x'\sin x+x(\sin x)'$
  $=\sin x+x\cos x$
- $\left(\frac{\sin x}{x^2}\right)'=\frac{x^2\cos x-2x\sin x}{x^4}$
- $(\sin(\cos x))'=\frac{d(\sin u)}{du}\cdot\frac{du}{dx}$ *(let $u=\cos x$)*
  $=\cos u\cdot(-\sin x)$
  $=\cos(\cos x)\cdot(-\sin x)$

Integration 積分

- **Integration**: Finding an antiderivative of a function **積分**：求函數的反導函數
- Examples:
  - $F(x) = x^2 \Rightarrow F'(x) = 2x$, and hence x^2 is an antiderivative of $2x$
  - $G(x) = x + \sin x + 3 \Rightarrow G'(x) = 1 + \cos x$,
    and hence $x + \sin x + 3$ is an antiderivative of $1 + \cos x$
- Remark: Antiderivatives are not unique! 注意：反導函數並非唯一的！
  e.g. for $F(x) = x^2 + 1$ and $G(x) = x^2 + 3$, $F'(x) = 2x = G'(x)$
- We usually write the integral as 我們通常將積分寫成以下形式：

$$\int f(x)\, dx = F(x) + C$$

where C is a constant 其中 C 為任意常數。

Integration rules and results 積分法則及結果

Integrals of common functions 常見函數的積分：

- $\int 0 \, dx = C$
- $\int ax^{a-1} = x^a + C$
- $\int e^x = e^x + C$
- $\int \frac{1}{x} dx = \ln x + C$ for $x > 0$
- $\int \cos x \, dx = \sin x + C$
- $\int -\sin x \, dx = \cos x + C$
- $\int \sec^2 x \, dx = \tan^2 x + C$

Integration rules and results 積分法則及結果

Integration rules 積分法則:

- $\int (u+v)\,dx = \int u\,dx + \int v\,dx$
- $\int \big(ku(x)\big)\,dx = k\int u(x)\,dx$
- $\int f\big(g(x)\big)g'(x)\,dx = \int f(u)du$
  where $u = g(x)$
  (integration by substitution 換元積分法)
- $\int u\,dv = uv - \int v\,du$
  (integration by parts 分部積分法)

- $\int (2x+1)dx = x^2 + x + C$
- $\int 2x^2\,dx = 2\int x^2 dx = \frac{2x^3}{3} + C$
- $\int \sin(\cos x)\sin x\,dx$
  $= -\int \sin(\cos x)\,d(\cos x)$
  $= -\int \sin(u)\,du$ (let $u = \cos x$)
  $= \cos u + C = \cos(\cos x) + C$
- $\int t\cos t\ dt$
  $= \int t\ d(\sin t)$
  $= t\sin t - \int \sin t\,dt = t\sin t + \cos t + C$

Exercises

1. Find the derivatives of the following functions:

(a) $y = f(x) = x^3 e^x$

(b) $y = f(x) = \sin(x + \cos x)$

(c) $y = f(x) = x^x$ where $x > 0$ (Hint: use logarithm)

2. Find the following integrals:

(a) $\int x \cos\left(x^2 + 1\right)\, dx$

(b) $\int \ln x \, dx$ where $x > 0$ (Hint: use integration by parts)

Multivariable calculus 多元微積分

- In many situations, there are more than one variable!
  在許多實際情況中，函數可能依賴一個以上的變數！

- Examples:
  - The sale of a product may depend on 某產品的銷售額可能同時取決於:
    - Population, raw material cost, time, weather, …
      人口、原料成本、時間、天氣……
  - $F(x, y, z) = x^2 + 2y^4 + 3xyz + \cos(xyz)$

- For multivariable functions, we want to 對於多元函數，我們希望能夠 :
  - Study how a change in one factor will affect the function value
    研究當其中一個因素改變時，對整體函數值的影響
  - The minimum/maximum value of the function (similar to the single-variable case) 求取函數的最大值或最小值（原理類似單變數的情況）

Partial derivatives 偏微分

- We can calculate the partial derivatives of a function with respect to one of the variables 對於多元函數，我們可以計算函數對其中一個變數的偏導數
  - Treat all other variables as constant 將其他變數視為常數
  - Follow the usual differentiation rules 按照一般的微分法則進行求導

- Example: For $f(x, y) = x^2 + \sin(xy) + y^4$, we have
  - Partial derivative of f w.r.t. x 對 x 的偏導數:

$$\frac{\partial f}{\partial x} = 2x + y\cos(xy) + 0 = 2x + y\cos(xy)$$

  - Partial derivative of f w.r.t. y 對 y 的偏導數:

$$\frac{\partial f}{\partial y} = 0 + x\cos(xy) + 4y^3 = x\cos(xy) + 4y^3$$

Optimization 最優化

- Using partial derivatives, we can easily solve unconstrained optimization problems 我們可以利用偏導數，輕鬆求解無約束優化問題
  - Find the minimum (or maximum) of a differentiable function $f(x_1, x_2, \ldots, x_n)$ without any restrictions on $x_1, x_2, \ldots, x_n$
    即在沒有任何限制條件下，找出可微函數 $f(x_1, x_2, \ldots, x_n)$ 的最小值（或最大值）

- Analogous to the 1D case, the minimum/maximum points of the function must satisfy the condition that the gradient of f is 0 類似單變數情況，函數的極小值點或極大值點必須滿足梯度（Gradient）為零的條件：

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) = (0,0,\ldots,0)$$

  In other words, 也就是說

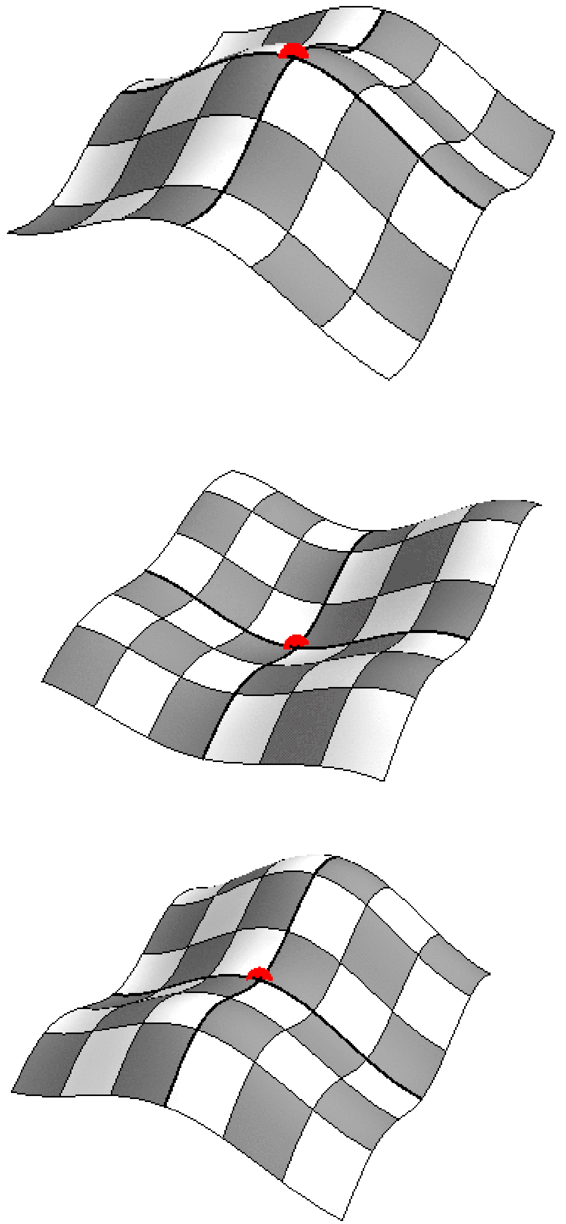
$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

Optimization 最優化

- To determine whether a critical point is a minimum/ maximum point, we have the following condition (similar to the second derivative test in the 1D case)
要確定某臨界點是否為最小值點或最大值點，我們可以使用以下條件（類似單變數的二階導數測試）

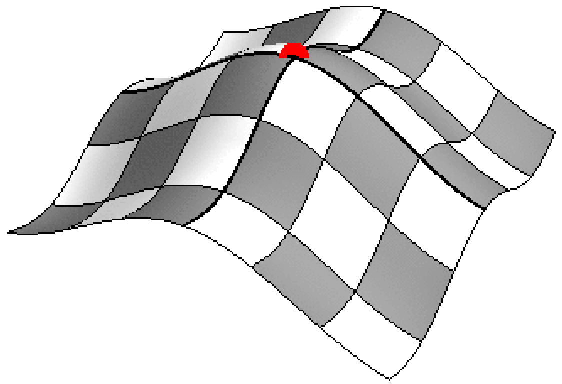
- Consider the **Hessian matrix** 考慮**海森矩陣**

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

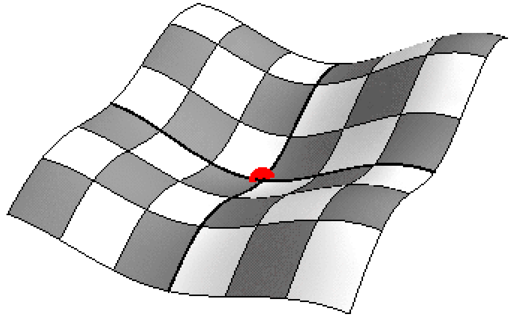


Optimization 最優化

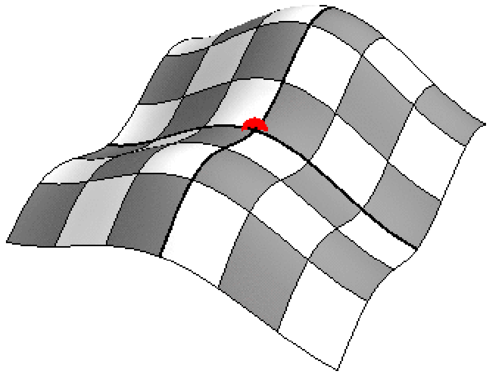
- If the Hessian is positive definite (equivalently, has all eigenvalues positive) at the point, then f attains a local minimum there.
  如果在該點處，海森矩陣為正定（等價於所有特徵值均為正），則函數 f 在該點取得局部最小值。



- If the Hessian is negative definite (equivalently, has all eigenvalues negative) at the point, then f attains a local maximum there.
  如果在該點處，海森矩陣為負定（等價於所有特徵值均為負），則函數 f 在該點取得局部最大值。



- If the Hessian has both positive and negative eigenvalues then the point is a saddle point for f.
  如果海森矩陣同時存在正的特徵值和負的特徵值，則該點為函數 f 的鞍



- If none of the above conditions holds, the test is inconclusive
  若以上情況均不符合，二階測試無法判斷

Optimization 最優化

- Example:

$$f(x,y) = x^2 + 3y^2 + 2xy$$

- Compute the gradient and critical point:

$$\frac{\partial f}{\partial x} = 2x + 2y, \qquad \frac{\partial f}{\partial y} = 2x + 6y$$

- For $\nabla f = 0$, we have $2x + 2y = 0$ and $2x + 6y = 0 \Rightarrow (x,y) = (0,0)$

- Compute Hessian matrix: $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix}$

- Eigenvalues of $H = 2\left(2+\sqrt{2}\right)$ and $2\left(2-\sqrt{2}\right)$, both positive
  $\Rightarrow (0,0)$ is a local minimum of f.

Exercises

Find all critical points of the following functions and determine their type (local minimum/local maximum/saddle):

(1) $f(x, y) = x^2 + y^2 + 2x + 4y + 5$

(2) $f(x, y) = -x^2 - 3y^2 + 4x + 6y$

(3) $f(x, y, z) = x^2 + y^2 - z^2 + 2xy + 4xz$

Solving optimization problems using Python

- We can easily solve optimization problems using **scipy.optimize** in Python

```
import numpy as np
from scipy.optimize import minimize

# define target functions 定義目標函數（input is a vector 輸入為向量 x = [x, y, z]）
def f(vars):
    x, y, z = vars
    return x**2 + 2*y**2 + 3*z**2 + 2*x*y + 2*x*z + 4*y*z

# initial guess (important!) 初始猜測值（很重要！）
initial_guess = [1.0, 1.0, 1.0]

# Perform optimization 執行優化
# method優化演算法: can choose among  'BFGS', 'L-BFGS-B', 'Nelder-Mead', 'SLSQP'
result = minimize(f, initial_guess, method='BFGS')

print("Minimum point location 最小值位置:", result.x)
print("Minimum value 最小值 f(x):", result.fun)
```

Constrained optimization problem 有約束優化問題

- We can also solve constrained optimization problem 有約束優化問題:
  - Minimize or maximize $f(x_1, x_2, \dots, x_n)$ subject to certain constraints
    - Can be equality or inequality constraints
  - Similar to the linear programming problem formulation last time, but the function and the constraints can all be nonlinear
- For example:

$$\min f(x, y, z) = x^2 - 2xy\exp(z) + z^4$$

subject to

$$\begin{cases} x + y \le 1 \\ x^2 + z^2 = 3 \end{cases}$$

Solving optimization problems using Python

- We can also utilize scipy.optimize.minimize to solve constrained optimization problem. To achieve this, we further prescribe constraints and bounds in the solver:

```
# Equality: x + y + z = 10
def eq_cons(x):
    return x[0] + x[1] + x[2] - 10

# Inequality:
cons = [
    {'type': 'eq',   'fun': eq_cons},                          # Equality: x + y + z = 10
    {'type': 'ineq', 'fun': lambda x: x[0] + x[1] - 6},  # x+y >= 6
    {'type': 'ineq', 'fun': lambda x: 20 - x[1]*x[2]}    # yz <= 20
]

# bounds
bounds = [(0, 8), (0, None), (0, None)]

result = minimize(objective, x0, method='SLSQP', bounds=bounds, constraints=cons)
```

Remarks on optimization

- We will usually write a minimization problem as

$$f^* = \min_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n)$$

  subject to certain constraints
  (similarly for maximization problem $f^* = \max_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n)$)
  - f^* represents the corresponding minimum/maximum function value
  - Note that many computational algorithms only focus on minimization problems. For maximization problems, we can simply solve their equivalent minimization problems $f^* = \min_{x_1, x_2, \ldots, x_n} -f(x_1, \ldots, x_n)$ (the actual maximum value is then $-f^*$).

- Also, if we want to talk about the input value (the "argument") that gives such minimum/maximum function values, we will use **argmin** (short for argument of the minimum) or **argmax** (short for argument of the maximum), i.e.,

$$(x_1^*, \ldots, x_n^*) = \operatorname*{argmin}_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n) \quad \text{or} \quad (x_1^*, \ldots, x_n^*) = \operatorname*{argmax}_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n)$$

What are differential equations? 甚麼是微分方程？

- A **differential equation (DE)** is an equation containing derivatives of a function
  微分方程（DE）是包含函數導數的方程式。

- If the function depends on one variable only (e.g. $y = f(x)$), we call the differential equation an **ordinary differential equation (ODE)**
  如果函數只依賴一個變數（例如 y=$f(x)$），我們稱此微分方程為**常微分方程(ODE)**。

- The **order** of a DE is the order of the highest derivative in the equation.
  微分方程的**階數**是指方程式中最高階導數的階數。

Differential Equation	Order	Unknown Function
$\frac{dy}{dx} = 4y$	1	$y(x)$
$y'' + 2y = 2x$	2	$y(x)$
$\frac{d^3y}{dt^3} - t\frac{dy}{dt} + t(y-1) = e^t$	3	$y(t)$

What are differential equations? 甚麼是微分方程？

- How to solve a given ODE to get a solution function $y = f(x)$?
- Examples:
  - $y' = x^2 \Rightarrow y = f(x) = \frac{x^3}{3} + C$
  - $y' + y = x^2 \Rightarrow y = ??$
  - $y'' + y' - y = \sin x \Rightarrow y = ??$
- In general, we need different techniques depending on the overall form and order of the ODE
- **Not all** differential equations can be solved analytically (i.e. we may not be able to find an explicit formula of the solution $y = f(x)$).
  - Computational tools will be very useful!

Separable differential equation 可分離微分方程

- If we have a differential equation of the form 如果我們有一個形式為

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Then we call it a separable differential equation.
的微分方程，則我們稱它為可分離微分方程。

Example:

$$\frac{dy}{dx} = xy^2 \quad ✓$$

$$\frac{dy}{dx} = \sin(xy) \quad ✗$$

$$\frac{dy}{dt} = \frac{\sin t}{\sqrt{y^2+1}} \quad ✓$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{y^2+t+1}} \quad ✗$$

$$\frac{dy}{dx} = e^{x+y} - 2e^x = e^x(e^y - 2) \quad ✓$$

$$\frac{dy}{dx} = x^{\sin y} + y \quad ✗$$

Separation of variables 分離變數法

- If the ODE is separable, we can rewrite $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ as

$$h(y)\frac{dy}{dx} = g(x)$$

Sometimes also written informally as
$$h(y)dy = g(x)dx$$

and integrate both sides:

$$\int h(y)\frac{dy}{dx}\,dx = \int g(x)\,dx$$

- By chain rule, this gives

$$\int h(y)\,dy = \int g(x)\,dx$$

- By handling both sides using standard integration techniques, we can solve the ODE and get

$$\boldsymbol{H(y) = G(x) + C}$$

Separation of variables 分離變數法

Solve the differential equation

$$\frac{dy}{dx} = \frac{2x^3}{y^2}.$$

Solution. The equation is rewritten as

$$y^2 dy = 2x^3 dx.$$

Integrate both sides,

$$\int y^2 dy = \int 2x^3 dx,$$

Hence the general solution is

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + C.$$

Separation of variables 分離變數法

- **Initial value problem (IVP):**
  A differential equation (not necessarily separable) together with an initial condition specifying the value of the function at a certain point.

Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2x^3}{y^2}, \\ y(0) = 1. \end{cases}$$

Solution. The general solution is

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + C.$$

Substituting the initial condition $y(0) = 1$ into the general solution yields $C = \frac{1}{3}$. Thus, the solution to the initial value problem is

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + \frac{1}{3}.$$

Exercises

1. Solve the following ODE for the function $y(x)$:

$$\frac{dy}{dx} = \frac{x^2}{\sin y}$$

2. Solve the initial value problem for the function $y(t)$:

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{t+1}{y} \\ y(0) = 2 \end{cases}$$

Modelling population growth

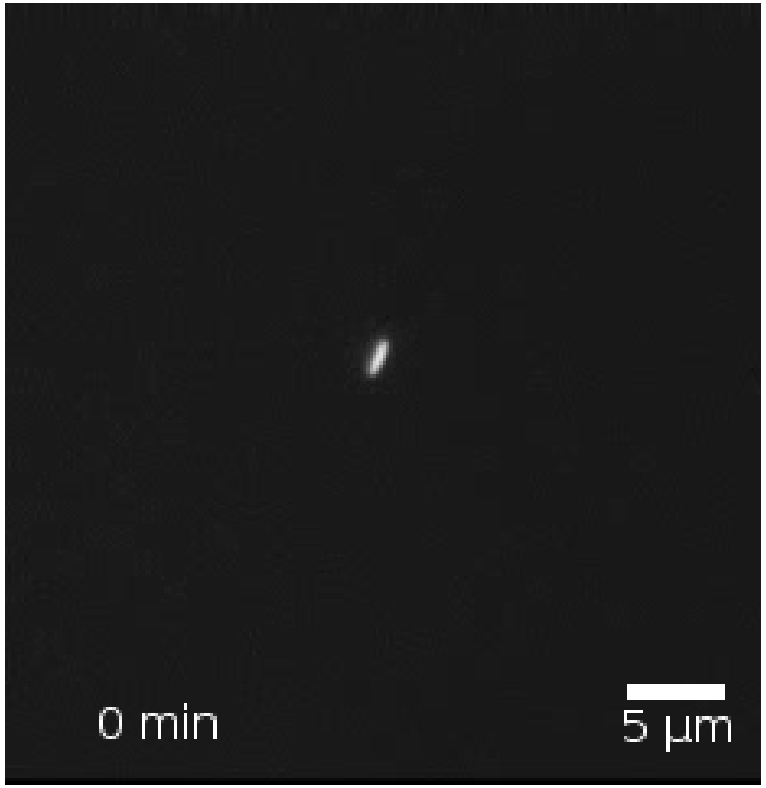
- The rate of change of the population is proportional to the current population

- **Model it using a differential equation:**

$$\frac{dy}{dt} = ky$$

  where k is a constant (the growth factor)

- Exercise:
  Solve the above differential equation with $k = 2$ and the initial condition $y(0) = 1$
  (i.e. it is given that at time $t = 0$, population = 1)



Modelling population growth

- The rate of change of the population is proportional to the current population
- **Model it using a differential equation:**

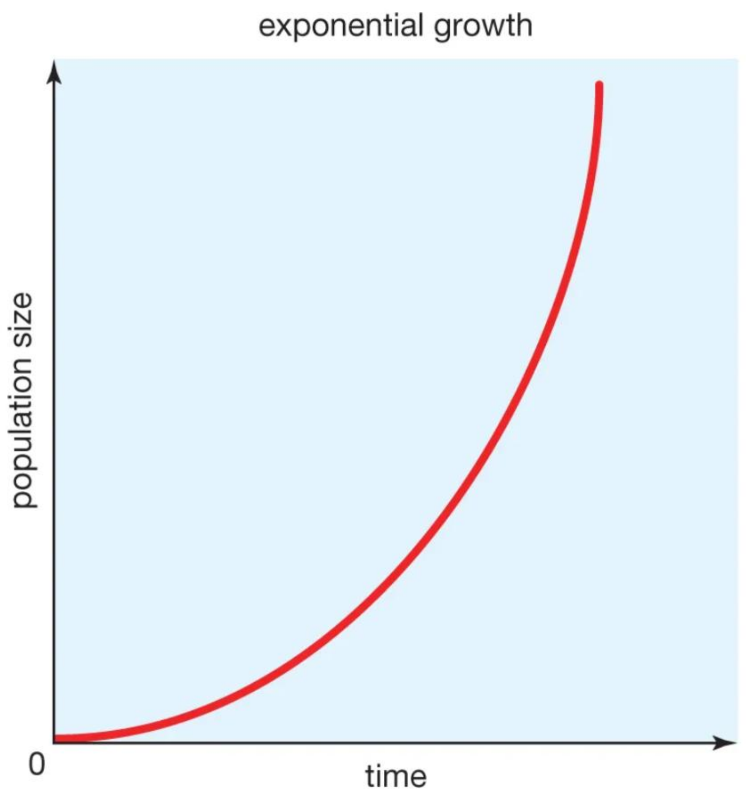
$$\frac{dy}{dt} = ky$$

where k is a constant (the growth factor)

- The general solution is given by:

$$y = Ce^{kt}$$

- Known as the **exponential model**



First-order linear ODE

- **What if we have some more complicated scenarios?**

e.g. $\frac{dy}{dt} = 2y - 100$

- The rate of change is partially proportional to y
- but there are also other factors (e.g. lack of resources, competitions, predators, …) that constantly decrease the rate of change

e.g. $\frac{dy}{dt} = 2y - 100 + \sin t$

- The rate of change is partially proportional to y
- but there are also other factors (e.g. lack of resources, competitions, predators, …) that constantly decrease the rate of change
- The rate of change will also depend on t periodically

First-order linear ODE: method of integrating factor

- Consider

$$\frac{dy}{dx} + p(x)y = q(x)$$

  - Only involving first derivative $\frac{dy}{dx}$
  - $p(x)$, $q(x)$ can be any functions of x
  - No terms like $\sin(y')$, yy', etc.

e.g. $\frac{dy}{dt} = 2y - 100 \quad \Rightarrow \quad \frac{dy}{dt} - 2y = -100$

e.g. $\frac{dy}{dt} = 2y - 100 + \sin t \quad \Rightarrow \quad \frac{dy}{dt} - 2y = -100 + \sin t$

First-order linear ODE: method of integrating factor

- Define the integrating factor for $\frac{dy}{dx} + p(x)y = q(x)$ as the following function:

$$\mu(x) = e^{\int p(x)dx}$$

- Then we have:

$$(\mu(x)y)' = \mu(x)q(x)$$

- Integrating both sides:

$$\boldsymbol{\mu(x)y = H(x) + C}$$

- Multiplying this factor can help solve the DE magically!

Explanation:

$(\mu(x)y)' = \mu'(x)y + \mu(x)y'$ (product rule)

$= \left(e^{\int p(x)dx}\right)' y + \mu(x)y'$ (by definition)

$= \left(p(x)e^{\int p(x)dx}\right) y + \mu(x)y'$ (chain rule and property of exp)

$= p(x)\mu(x)y + \mu(x)y'$ (by definition)

$= \mu(x)(y' + p(x)y)$ (grouping)

$= \mu(x)q(x)$ (by the DE)

First-order linear ODE: method of integrating factor

Solve

$$\frac{dy}{dx} - y = e^{3x}.$$

Solution. It is a first order linear equation with $p(x) = -1$ and $q(x) = e^{3x}$. Let

$$\mu = e^{\int p(x)dx} = e^{-x}.$$

(Caution: it should be e^{-x+C}, but we choose $C = 0$.)

Multiply the equation by μ,

$$\frac{d}{dx}(e^{-x}y) = e^{-x}e^{3x} = e^{2x}.$$

So

$$e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

(Caution: this C cannot be omitted.)

Hence, the general solution is

$$y = e^{x}(\frac{1}{2}e^{2x} + C).$$

Exercises

1. Solve

$$y' + y = xe^{-x} + 1$$

2. Solve the initial value problem for $y(t)$ with $t > 0$:

$$\begin{cases} ty' + 2y = \dfrac{\sin t}{t} \\ y\left(\dfrac{\pi}{2}\right) = 0 \end{cases}$$

(Hint: Rewrite the ODE as an appropriate form first.)

Solving first-order ODE in Python

- So far we know how to solve:
  - $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ (separable)
  - $\frac{dy}{dx} + p(x)y = q(x)$ (some of the first-order ODEs, not all!)
- What about the other first-order ODEs?
  - e.g. $\frac{dy}{dx} = \sin(\cos(xy)) + e^{x}y^{2} + \frac{1}{x}$
- In general, we have to use computer to solve them numerically
  - Principle: discrete approximation of the derivative (afternoon session)
  - Computational tool in Python: odeint

Higher-order ODEs

- Higher-order ODEs:
  - Differential equations involving higher-order derivatives $(y'', y''', \dots)$

- Example:
  - 2nd-order ODE with constant coefficients:

$$y'' - 2y' + 4y = 0$$

  - 3rd-order ODE with nonconstant coefficients:

$$(\sin t)y''' + 2y'' + t^3 y' - ty = 4$$

  - Newton's second law $F = ma = mx''$

- How to solve (some of) them?

Higher-order ODEs: method of characteristic equation

- Consider the following 2nd-order ODE with constant coefficients

$$ay'' + by' + cy = 0$$

where a, b, c are constants and $a \neq 0$

- If we try to plug in $y = e^{rx}$ (where r is a constant), then we have

$$a(e^{rx})'' + b(e^{rx})' + ce^{rx} = 0$$
$$a\left(r^2 e^{rx}\right) + b(re^{rx}) + ce^{rx} = 0$$
$$\left(ar^2 + br + c\right)e^{rx} = 0$$

- Since $e^{rx} \neq 0$, we have $ar^2 + br + c = 0$

- In other words, $y = e^{rx}$ with r satisfying $ar^2 + br + c = 0$ will be a solution to the ODE

Higher-order ODEs: method of characteristic equation

- Therefore, to solve $ay'' + by' + cy = 0$, we only need to solve the characteristic equation $ar^2 + br + c = 0$ and find all its roots $r = r_1,\ r = r_2$

- Case 1: If we get two distinct real roots r_1, r_2 (i.e. $b^2 - 4ac > 0$), then the general solution is
$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$
where C_1, C_2 are constants

- Case 2: If we get two repeated real roots r (i.e. $b^2 - 4ac = 0$), then the general solution is
$$y(x) = C_1 e^{rx} + C_2 x e^{rx}$$

- Case 3: If we get two distinct complex roots $r = \alpha \pm \beta i$ where $i^2 = -1$ (i.e. $b^2 - 4ac < 0$), then the general solution is
$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Higher-order ODEs: method of characteristic equation

- Example: Consider the second-order ODE for $y(x)$:
$$y'' - y' - 6y = 0$$
The characteristic equation is
$$r^2 - r - 6 = 0$$
$$(r-3)(r+2) = 0$$
$$r = 3, -2$$
Therefore, the general solution to the ODE is
$$y = C_1 e^{3x} + C_2 e^{-2x}$$

- Remark: The same approach also works for higher-order ODEs with constant coefficient!
  - Example: For $y''' + 2y'' - 5y' - 6y = 0$, solve $r^3 + 2r^2 - 5r - 6 = 0$ and get $y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x}$

Exercise

Solve the initial value problem for the function $y(t)$:

$$\begin{cases} 2y'' - 3y' + y = 0 \\ y(0) = 2 \\ y'(0) = \dfrac{1}{2} \end{cases}$$

Advanced topics

- Solving systems of ODEs
- Computational tools (see Python notebooks) for solving:
  - Systems of ODEs
  - Higher-order ODEs
- More modelling examples
- Theoretical analysis of ODE systems

Mathematical Modelling @ CUHK Mathematics:
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Thank you!